Abstract

In this Circles, we play and describe the game of Nim and some of its friends. In German, the word *nimm!* is an excited form of the verb to take. For example to tell someone to take "it all" you would say *Nimm alles!* and to tell someone to "keep themselves together" you would say *Nimm dich zusammen!*. As you can guess, in this Circles we will talk about games where you take things. Hopefully when we are done, you will be able to beat all of your friends at these games whenever you want, thus "taking the title Gamemaster" or *Nimm den Titel Meister der Spiele!*.

1 Definitions

Definition 1 (Game). *For us, a **game** will be any series of alternating moves between two players where one player must win.*

Definition 2 (States). *Any particular position that the board will can be in will be called a* state*. The first position will be called the* initial state *and the last position will be called the* final state*.*

Definition 3 (Impartial Game). *A game is called **impartial** if the possible moves that a player can take only depends on the current state of the game and not on which player they are. All of the games that we will look at are impartial. There are many games which are not impartial (called* partisan*), like chess and checkers. In chess, only player 1 can move the white pieces, and so the game is not impartial.*

Definition 4 (Normal Play). *A game is finished when it has reached one of the* final positions *(we will always only have one final position). In* normal play*, the player that puts the game into the final position is the winner.*

Definition 5 (Misére Play). *In* misére play*, the player that puts the game into the final position is the loser.*

2 Game of 21

The first friend of Nim that we will play is called the Game of 21. In this game there is a pile of 21 stones and players take 1,2, or 3 stones in turns. The winner is the player who takes the last stone. This kind of play is called normal play. You can also play misére where the person to take the last stone loses. Our job will be to take this simple game and determine if we can always determine who will win from the beginning of the game. Try playing the game a few times to see what it is like. Try both normal play and misére play.

2.1 Smaller Games

Play the Game of 21 with fewer stones. Start with 1, then with 2, then 3, and so on. Play each game 5 times and record who wins (the first player or the second player) for both normal and misére play.
Here is a table for recording the winners
What positions do you want to put your opponent in if you can?
Is there a pattern to who wins and who loses?
Can you make any statements about beginning positions that give a winner?
What would you do if it were instead the Game of 22 and you went first?

Table 1: Game of N Table

N	Normal Play Winner Player 1	Normal Play Winner Player 2	Misére Play Winner Player 1	Misére Play Winner Player 2
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				
21				

3 Subtract a Square

The second friend of Nim that we will play is called Subtract a Square. In this game there is a pile of stones and players remove a number of stones in turns. The number of stones taken has to be a square, so 1, 4, 9, 16, 25, etc. The game can be played both normal and misére and with any number of stones. Try playing the game with 51 stones under both playing styles.

3.1 Smaller Games

Play Subtract a square with fewer stones. Play each game 5 times and record who wins (the first player or the second player) for both normal and misére play.

Here is a table for recording the winners

What positions do you want to put your opponent in if you can?

Is there a pattern to who wins and who loses?

Can you make any statements about beginning positions that give a winner?

What would you do if the beginning number was 446?

Table 2: Subtract a Square Table

	Normal Play Winner		Misére Play Winner	
N	Player 1	Player 2	Player 1	Player 2
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
15				
19				
27				
31				
35				
41				
43				
45				
48				
50				

4 Game of Nim

Now for the more complicated game, the Game of Nim. Nim is played with multiple piles of stones. Each pile has a number of stones. One a given turn a player has to take at least one stone and can take a many stones from any one pile as they like. Try playing the game with beginning piles of 6,10, and 15 stones under both normal and misére play.

4.1 Smaller Games of Nim

Play Subtract a square with fewer stones. Play each game 5 times and record who wins (the first player or the second player) for both normal and misére play.
Here is a table for recording the winners
Is there a pattern to who wins and who loses?
Is there a way to always win if the game begins as (n, n, k) for any numbers n and k?
If the game began as $(1, 2, 3, 4)$, who do you think wins? Play this game a few times and see what conclusions you can make?

Table 3: Game of Nim Table

N	Normal Play Winner Player 1	Normal Play Winner Player 2	Misére Play Winner Player 1	Misére Play Winner Player 2
$(1,1,1)$				
$(1,1,2)$				
$(1,1,3)$				
$(1,2,2)$				
$(1,1,4)$				
$(1,2,3)$				
$(2,2,2)$				
$(1,1,5)$				
$(1,2,4)$				
$(1,3,3)$				
$(2,2,3)$				
$(1,1,6)$				
$(1,2,5)$				
$(1,3,4)$				
$(2,2,4)$				
$(2,3,3)$				
$(1,1,7)$				
$(1,2,6)$				
$(1,3,5)$				
$(1,4,4)$				
$(2,2,5)$				
$(2,3,4)$				

5 Theory of Impartial Games

We want to characterize the states of the game as either winning of losing. For a given state, we will call it a *winning position* (W) if you can always win from it and we will call the state a *losing position* (L) if you cannot win from it. To start characterizing the states of the game, think about starting with simple states and moving to more complicated states.
For $N = 1, 2, \ldots, 21$, can you characterize each state as winning or losing in the Game of N?
For $N = 1, 2, \ldots, 21$, can you characterize each state as winning or losing in Subtract a Square?
Do you think that there is a way to characterize winning and losing states for the Game of Nim?

5.1 How to Win at Games

The general strategy for winning a game is like this:
If you can always force your opponent into a losing position, then you will always win!
A winning position is one where you can always put your opponent into a losing position.
And so a losing position has to be a state where you *cannot* put your opponent into a losing position (you can only put your opponent into a winning position).
If we can characterize all of the losing positions of a game, then the rest of the positions have to be winning positions (someone has to win).
It turns out that this can always be done for normal play and for misére games that are not too crazy (these are called tame).
So, we will focus on normal play in our games.
This is sometimes very hard to think about, but can be made very easy for the games that we have seen and some memory will help you to always win at these games under normal play.

5.2 Game of N with Normal Play

In order to determine who will win at the Game of N, we can analyze all of the games that are simpler than N because after the first move in the game of N, we are left with a game with less that N stones and that leads to a game that we have already analyzed. Let's start analyzing the states. You always go first and I always go second.

$N = 1$, W You take the stone and win!

$N = 2$, W You take both stones and win!

$N = 3$, W You take all three stones and win!

$N = 4$, L You have to take 1,2, or 3 stones. This leaves me with 3,2, or 1 stones (respectively) and I win! So, you lose, sorry.

$N = 5$, W You have to take 1,2, or 3 stones. This leaves me with 4,3, or 2 stones (respectively). If I have 4, then I am in a losing state, otherwise you put me in a winning state (3 or 2 stones). So, you leave me with 4 stones and win!

$N = 6$, W You have to take 1,2, or 3 stones. This leaves me with 5,4, or 3 stones (respectively). If I have 4, then I am in a losing state, otherwise you put me in a winning state (5 or 3 stones). So, you leave me with 4 stones and win!

$N = 7$, W You have to take 1,2, or 3 stones. This leaves me with 6,5, or 4 stones (respectively). If I have 4, then I am in a losing state, otherwise you put me in a winning state (6 or 5 stones). So, you leave me with 4 stones and win!

$N = 8$, L You have to take 1,2, or 3 stones. This leaves me with 7,6, or 5 stones (respectively). All of these are winning positions, but now I am in them, and so you have to lose, sorry.

Do you see a pattern to the Game of N?
If so, who will win the Game of 21? Who would win if the game had 44 stones? What about 105?
Can you make up a game that is like the game of N, but where you take away 7 stones instead of 4? What would determine the winner of that game?
Can you make a rule for always winning under misére play?

5.3 Subtract a Square with Normal Play

In order to determine who will win at Subtract a Square starting at the number N, we can analyze all of the games that are simpler than N because after the first move in the, we are left with a game with less that N stones and that leads to a game that we have already analyzed. Let's start analyzing the positions. You always go first and I always go second.

$N = 1$, W You take the stone and win!

$N = 2$, L You can only take 1 stone, and this leaves me with a win. Sorry, you lose.

$N = 3$, W You can only take 1 stone and this leave me with 2, a losing position. You win!

$N = 4$, W You take all of the stones and win!

$N = 5$, L You can take 1 or 4 stones. Unfortunately this leave me with 4 or 1 stones and I win.

$N = 6$, W You can take 1 or 4 stones. This leaves me with 5 or 2 stones, both losing positions. You win!

$N = 7$, L You can take 1 or 4 stones. Unfortunately this leave me with 6 or 3 stones and I win.

$N = 8$, W You can take 1 or 4 stones. This leaves me with 7 or 4 stones. If you leave me with 7, then I lose and you win!

$N = 9$, W You take all 9 stones and win.

We can keep going forever.
The important thing to characterize is the losing positions, as any other position is a winner.
I will list the first few losing positions $2, 5, 7, 10, 12, 15$.
Can you name the next 5 losing positions?
Is there any emergent pattern?
Can you make a similar rule for states in misére play?

5.4 Game of Nim

We have already seen a game with a simple rule and a game for which you would have to memorize the sequence of losing states. Both games are fairly easy to describe and the rules for winning are quite different. Though Nim seems like a much more complicated game, it will have a very simple rule for determining who will win.
What if the game begins as $(0, n, n)$?
In order to characterize Nim, we need some more mathematical machinery. Before that is introduced, see if you can characterize the states in the Nim table.

5.4.1 Base 2 Representation

Take any integer greater than 0. It can be represented uniquely as sequence of 0s and 1s that capture whether a given power of 2 divides the number. The number n is written as

$$n = n_0 2^0 + n_2 2^1 + n_3 2^3 + n_4 2^4 + \cdots \text{ where } n_i \text{ is 0 or 1}$$

If n_K is the largest such n_k, then the number is written as $n = n_K n_{K-1} \cdots n_3 n_2 n_1 n_0$. Here are the first few integers written in binary

$$\begin{aligned}
1 &= 1 \\
2 &= 10 \\
3 &= 11 \\
4 &= 100 \\
5 &= 101 \\
6 &= 110 \\
7 &= 111 \\
8 &= 1000 \\
9 &= 1001
\end{aligned}$$

What is 12 represented in base 2?
What is 16 represented in base 2?
What is 28 represented in base 2?

5.4.2 The $\oplus$ Operation

When we add numbers in binary, we can add them like normal numbers OR treat the binary representation as a sequence of switches. Each place holder (power of 2) represents a switch and the sum of two base 2 numbers is number in base 2 that represents which switches remain turned on. For example, $3 = 11$ and so the number 3 turns on the first two switches. If we add 3 and 1 using $\oplus$, we get $3 \oplus 1 = 11 \oplus 01 = 10 = 2$. In each place, if we see 2 ones, the switch is turned on and then back off. In each place where we see a 0 and a 1, the switch remains turned on. Essentially, each one represents flipping the switch to the position it is not in.

Here are some additions

$$
\begin{aligned}
1 \oplus 1 &= 1 \oplus 1 = 0 = 0 \\
1 \oplus 2 &= 01 \oplus 10 = 11 = 3 \\
1 \oplus 3 &= 01 \oplus 11 = 10 = 2 \\
1 \oplus 4 &= 01 \oplus 100 = 101 = 5 \\
2 \oplus 3 &= 10 \oplus 11 = 01 = 1 \\
5 \oplus 9 &= 0101 \oplus 1001 = 1100 = 12
\end{aligned}
$$

What is $12 \oplus 16$?
What is $12 \oplus 28$?
What is $16 \oplus 28$?
Is $\oplus$ commutative? associative?

5.4.3 Strategy for Nim

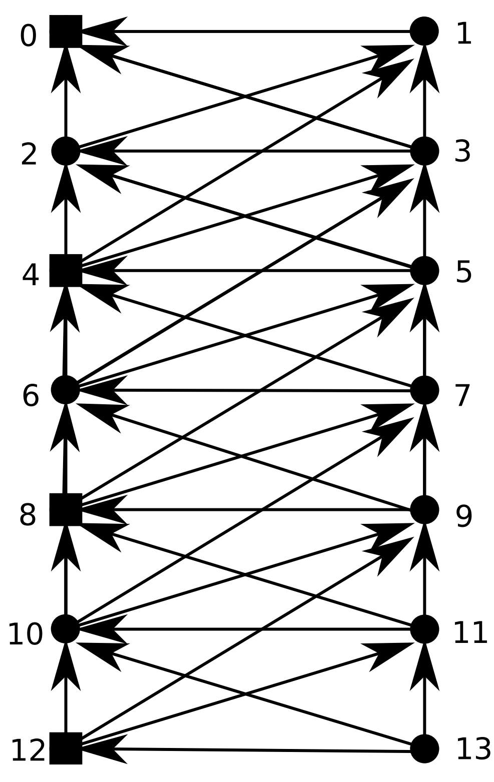
Now we come to the grand strategy. In Nim, represent each pile in base 2 and then add the numbers using $\oplus$. The losing states for normal play are the states where the sum (using $\oplus$) is 0. This is quite remarkable. It works for any number of piles and any sizes. Using this, we can determine if the player going first or second will win any given game of Nim.
Use this stategy to classigy the Nim games in the table.
Is the game $(3, 4, 5)$ in a winning or losing position?
Is the game $(3, 4, 12)$ in a winning or losing position?
Is the game $(1, 2, 3, 4, 5)$ in a winning or losing position?

Important things to prove:
From a 0 state, you can only go to a non-zero state.
From any non-zero state, you can go to the 0 state.
Can you prove these?

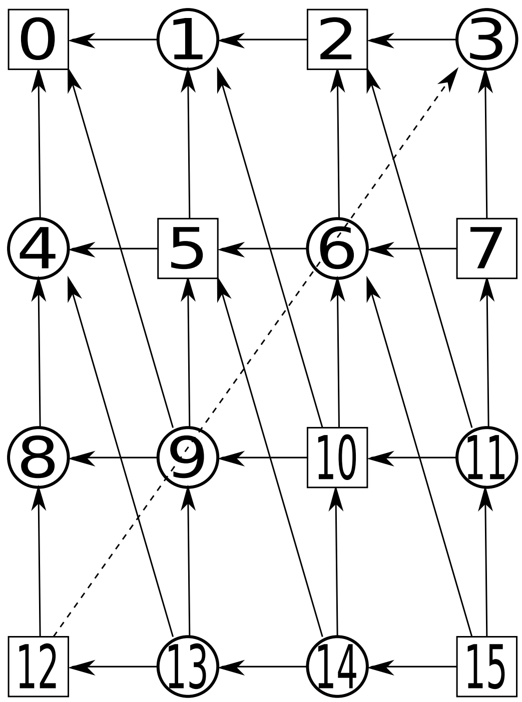
5.5 Graph of the Game

You can think of the states of the game as points in space, call them nodes. For each of our games under normal play, there is only one final state, the empty postion. Label this position with an L. Begin connecting the other states to this state by drawing lines between the final postion and the nodes that can lead to it in one move. Make the line an arrow pointing to the final state, representing that the final state can be reached in one turn. The node that lead to the final state are all winning nodes. Label them with a W. Keep doing this, connecting the nodes together. Any node that only leads to W nodes is an L node. Any node leading to at least one L node is a W node.
Here are some examples of graphs for the games we have seen. All of the losing nodes are boxes and the winning nodes are circles.

5.5.1 Game of N



5.5.2 Subtract a Square



5.5.3 Game of Nim

For this graph, we label a state by the piles in the state. So, a state with piles of size 1, 2, and 3 is labeled $(1, 2, 3)$. We treat the state $(1, 2, 3)$ as though it is the same as the state $(2, 1, 3)$ so that re-ordering of the piles does not matter. Can you draw a graph for the game of Nim with less than 5 total stones? less than 7 total stones? less than 9 total stones? Note that for the game of nim with 9 stones that there are 53 nodes!

6 Game of Grundy

This game is simlar to Nim, but begins with one pile of stones. On each turn, the players can split any pile into two unequal piles. The game ends when no pile can be split, so all of the remaining piles have 1 or 2 stones. This game can be played in both the normal and the misére style. Normal play can be analyzed in a similar manner to Nim.
Can you draw the graph of the game?
Can you find the way to analyze it?