KNOTS!

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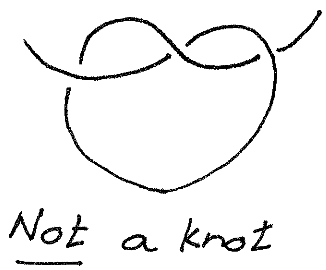
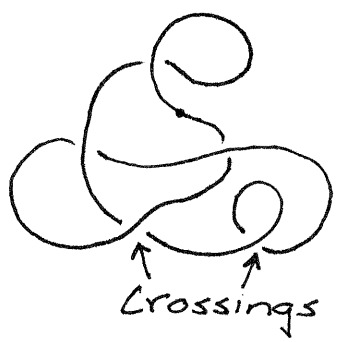
1. IT ALL COMES TO KNOT

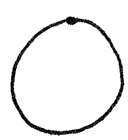
A piece of string (twisted in any old way) is called a knot only when its end points are stuck together. For example,



The knot diagram of a knot is what you get when the knot is flattened on to the paper (if an elephant sat on it!). For example,

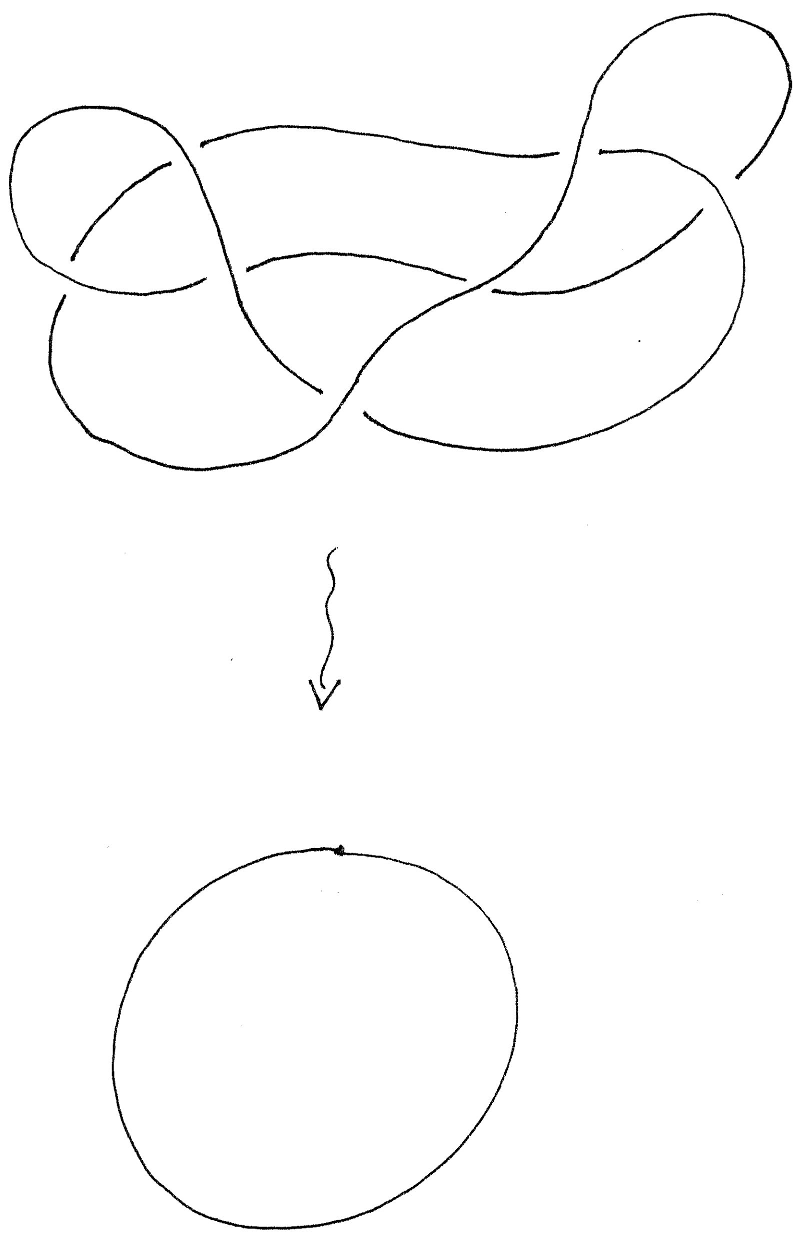


A knot is said to be the unknot if it has a circle as its diagram.

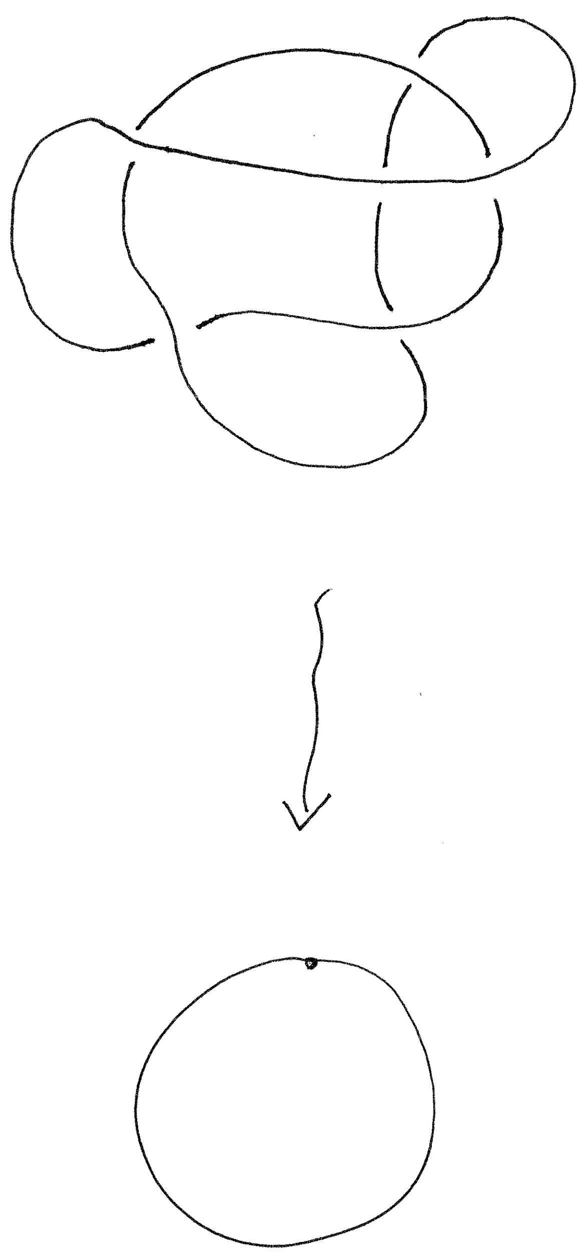


You can fit a knot to a given diagram by putting your thread on the diagram with the correct overcrossings and undercrossings and then clipping the ends of the thread together.

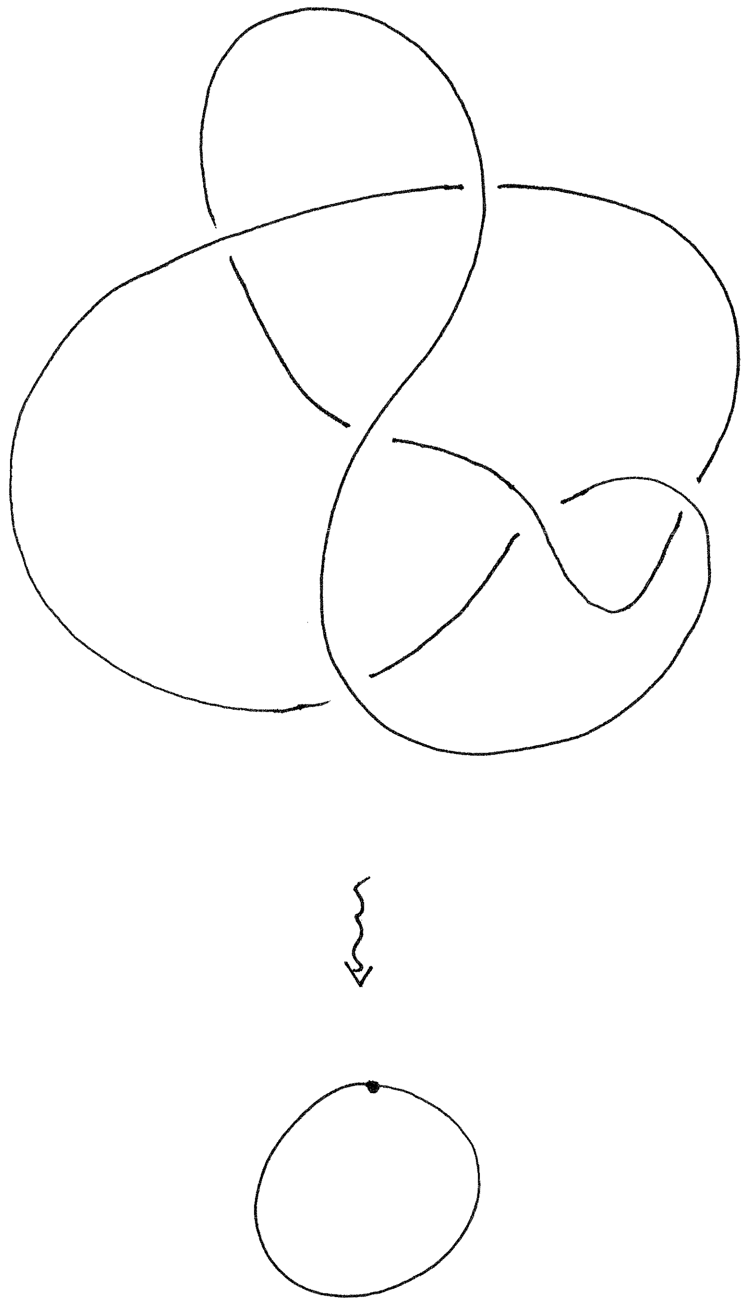
Fit a knot to the diagram given below and without unclipping the ends pull it about to try and unknot it (make it a circle), i.e, try to go from the diagram above to the diagram below without unclipping the ends.



Okay that was easy! Now just by looking at the following knot diagram try to see if the knot it makes can be unknotted or not. Once you've made a guess, fit a knot to the diagram and check.



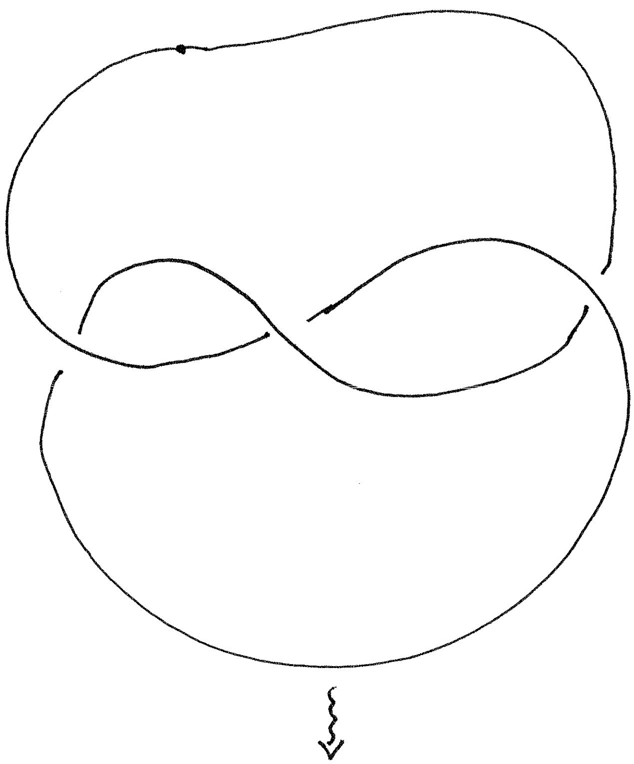
What about now ? Can you guess by just looking at this diagram if the knot it represents can be unknotted ? Fit a knot to the diagram and check your guess.



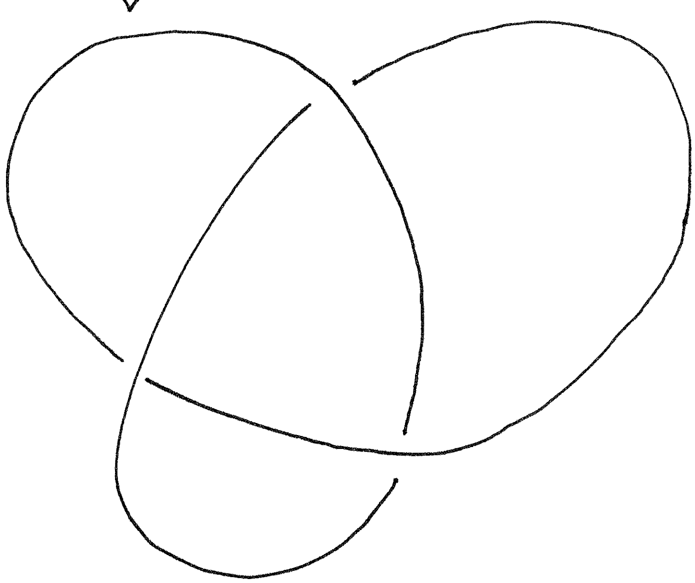
That's right... the last one cannot be unknotted, while the ones before it can. The moral here is that it is very difficult to determine if a knot is unknotted just by looking at its diagram.

Q.1 Is the knot represented by both these diagrams the same knot? In other words, if you fit a knot to Diagram a), can you pull it about (without unclipping) so that it can sit on Diagram b) (with the correct over and undercrossings)? First try to guess if it can be done and then check with your string.

a)



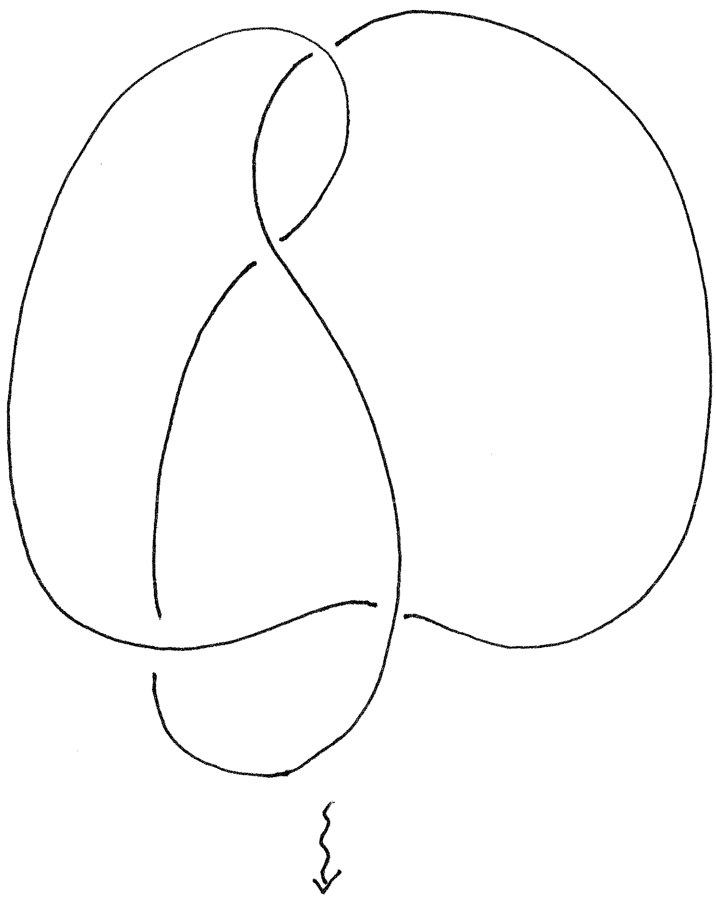
b)



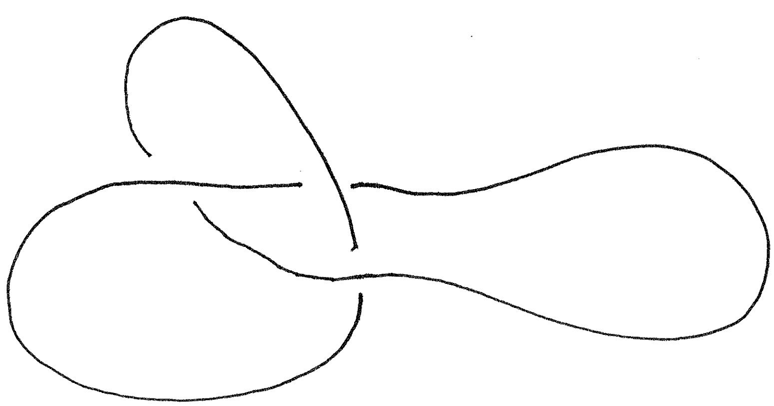
A.1 Yes you can, the knots that come from the two diagrams are the same.

Q.2 Again do the following two diagrams represent the same knot? First try to guess and then check with your string as before.

a)



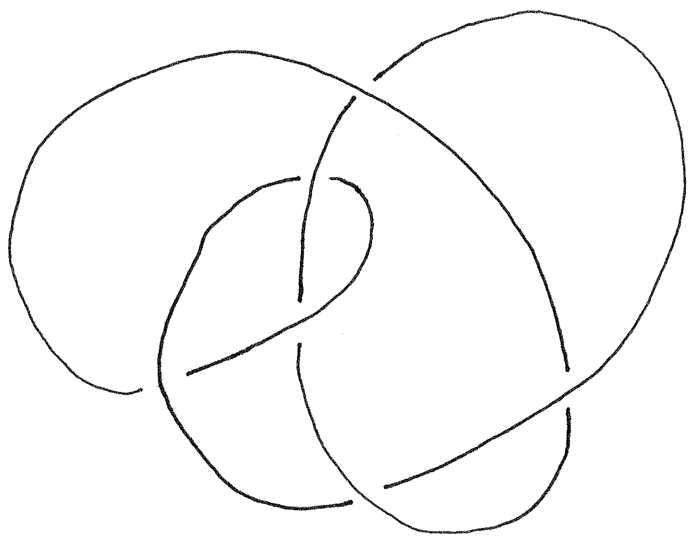
b)



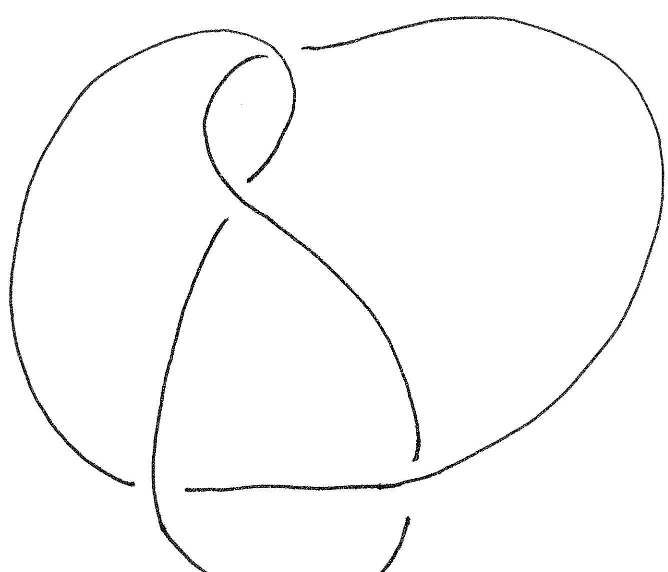
A2. Yes they are the same knot.

Q.3 Once more, do the following two diagrams represent the same knot? First try to guess and then check with your string.

a)



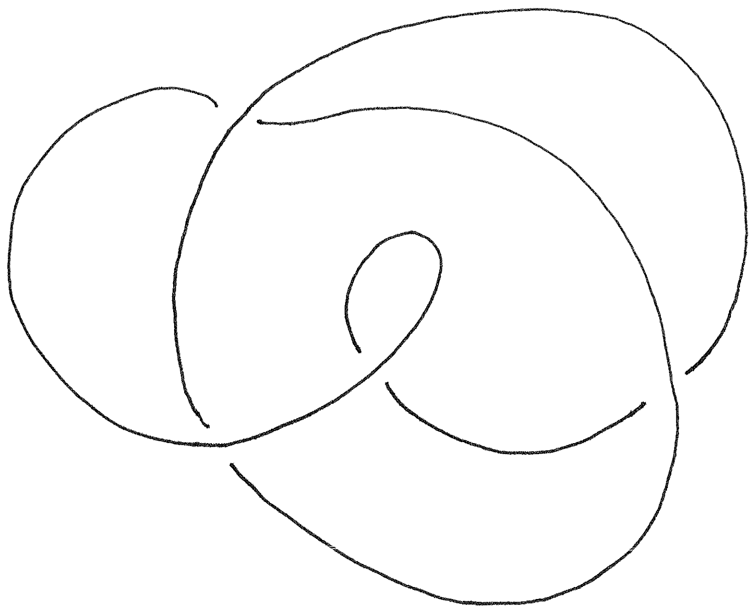
b)



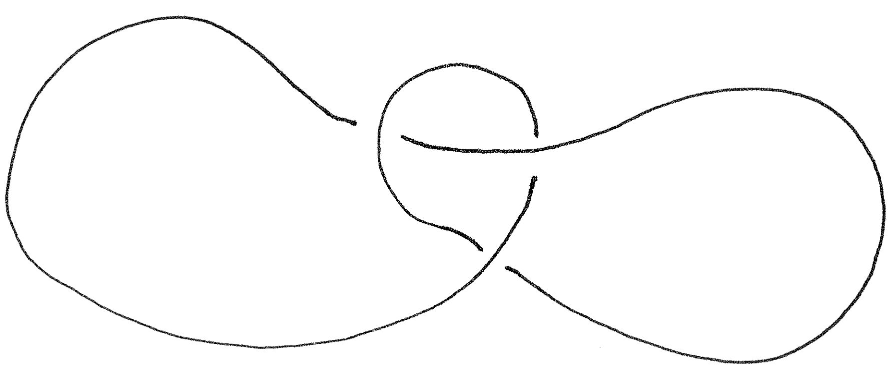
A3 No, They are not the same knot.

Q.4 And one more time, do the following two diagrams represent the same knot? First try to guess and then check with your string.

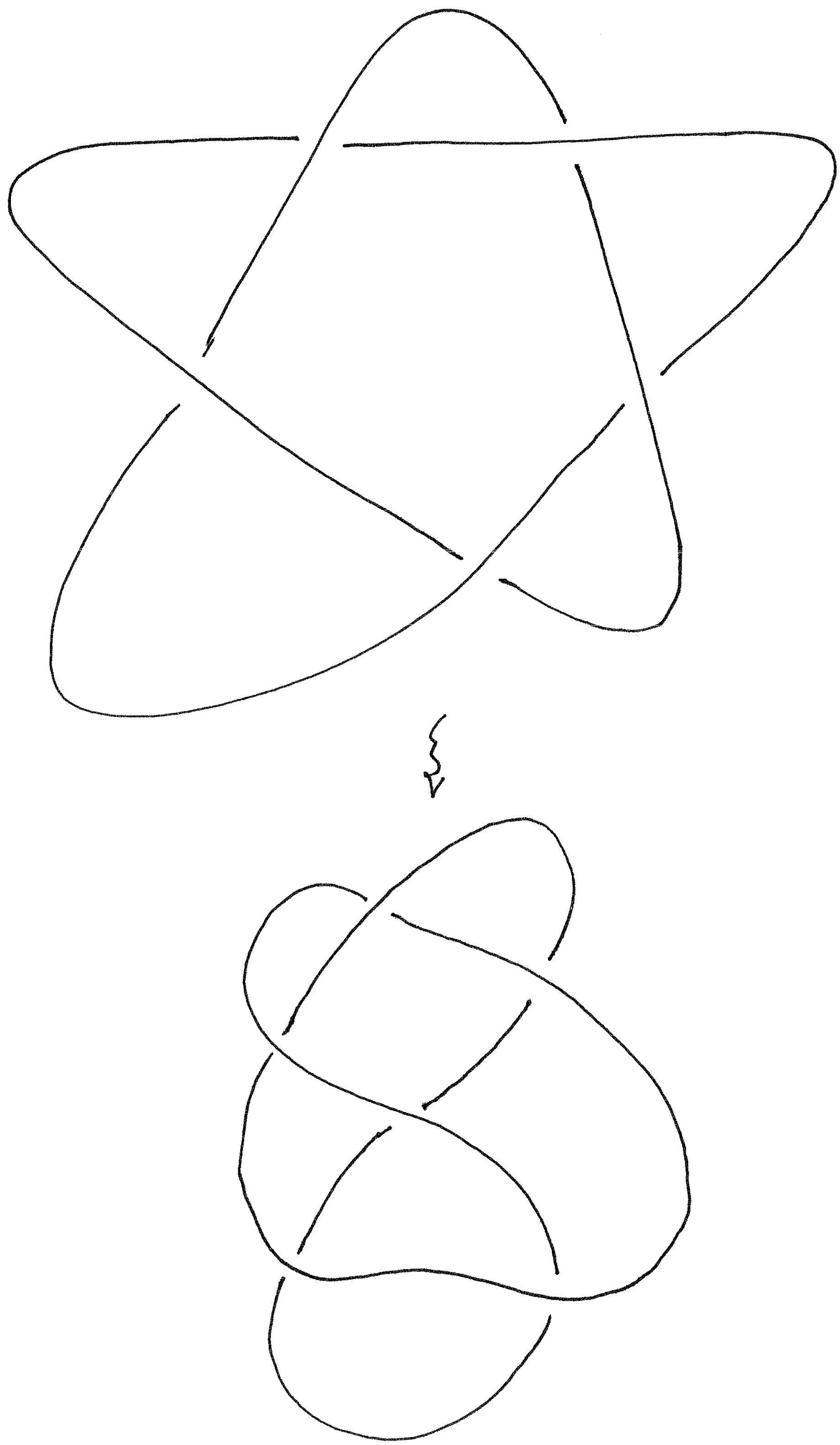
a)



b)



Q.5 Okay now one last time, do the following two diagrams represent the same knot ? First try to guess and then check with your string.



If you had twirled the string around correctly you would seen that in Q1, Q2 and Q4 the knot can be moved from Diagram a) to Diagram b) without unclipping or in other words, both the diagrams represent the same knot while the two Diagrams in Q3 and Q5 seem to represent different knots... but are they really really different? Were we not clever enough to twist them from a) to b) or are they genuinely different knots? Perhaps someone somewhere is smart enough to twist the knots to move from Diagram a) to b) in Q3 and Q5 although we could not do it. How do we prove beyond doubt that these diagrams do infact represent different knots? Well... turn the page to find out!

2. REIDEMEISTER (DANCE) MOVES

If you think of the various ways you twisted the knot to move from one diagram to the next as making the knot dance from one pose a) to pose b) then the real question is how do we make this 'dance' more systematic. In other words, can we insist that only a few dance moves are enough to move from one diagram of a knot to another?

Yes, we can! These moves are called the Reidemeister moves. There are three of them:

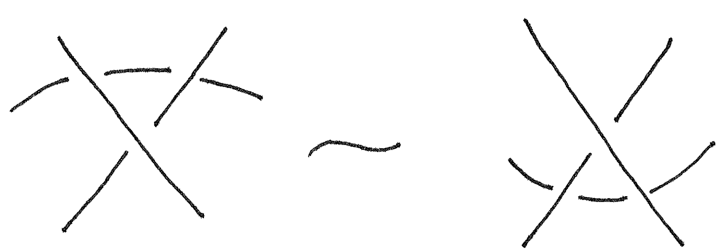
TYPE I. Removing or introducing kinks:

or

TYPE II. Pulling them apart or making them overlap:

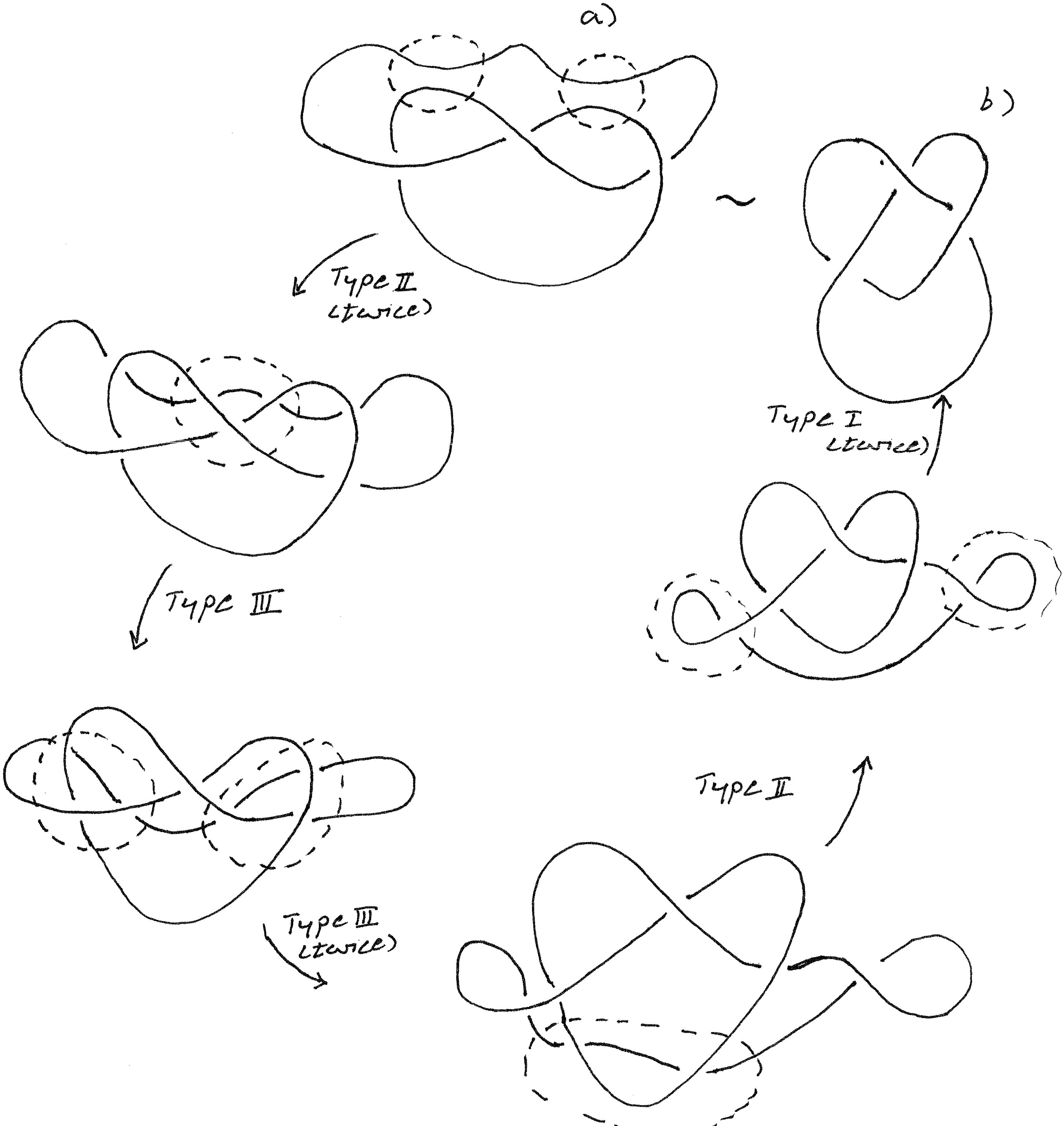
or

TYPE III. Moving the thread from under a crossing to over it or vice versa:

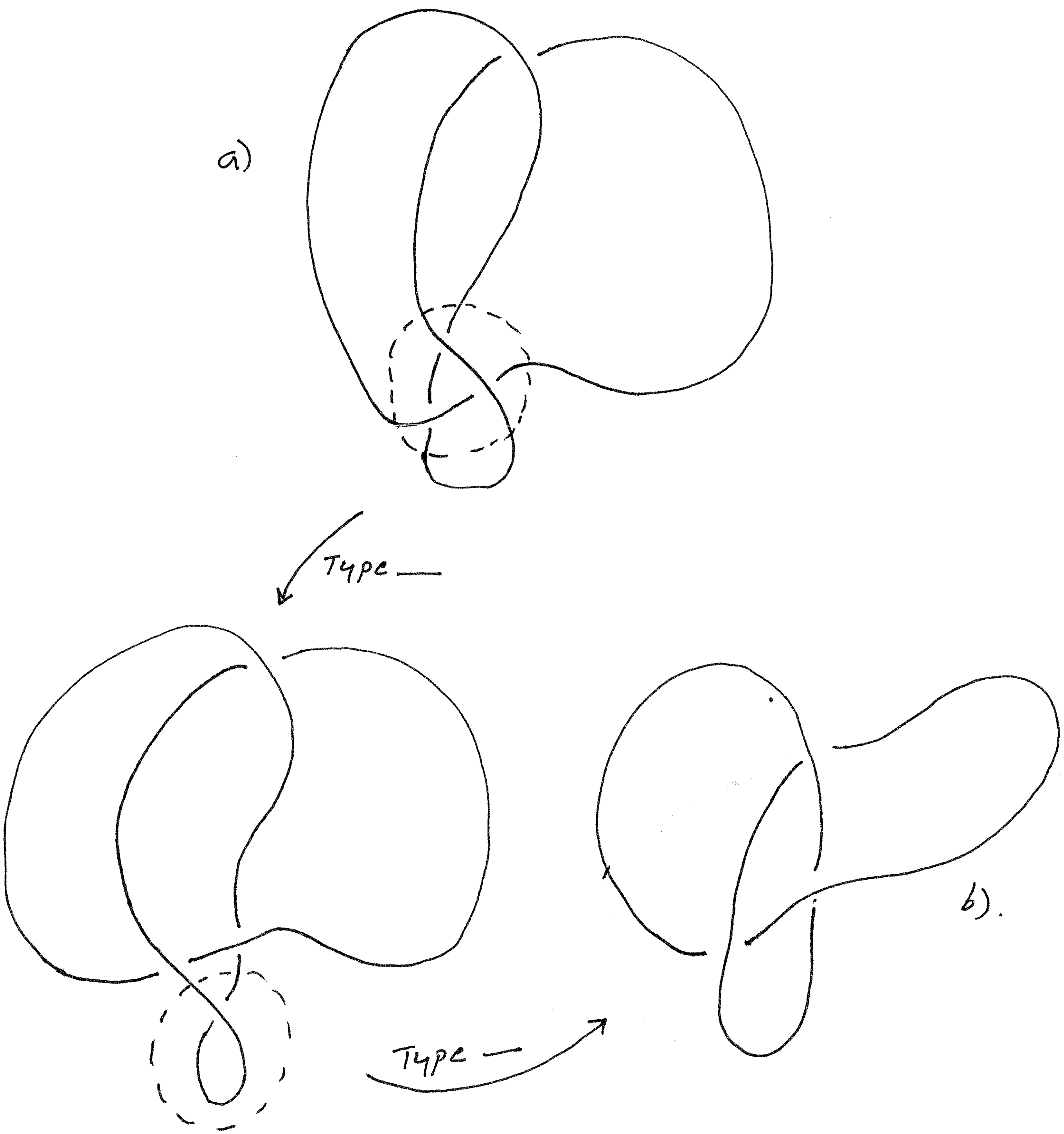


or

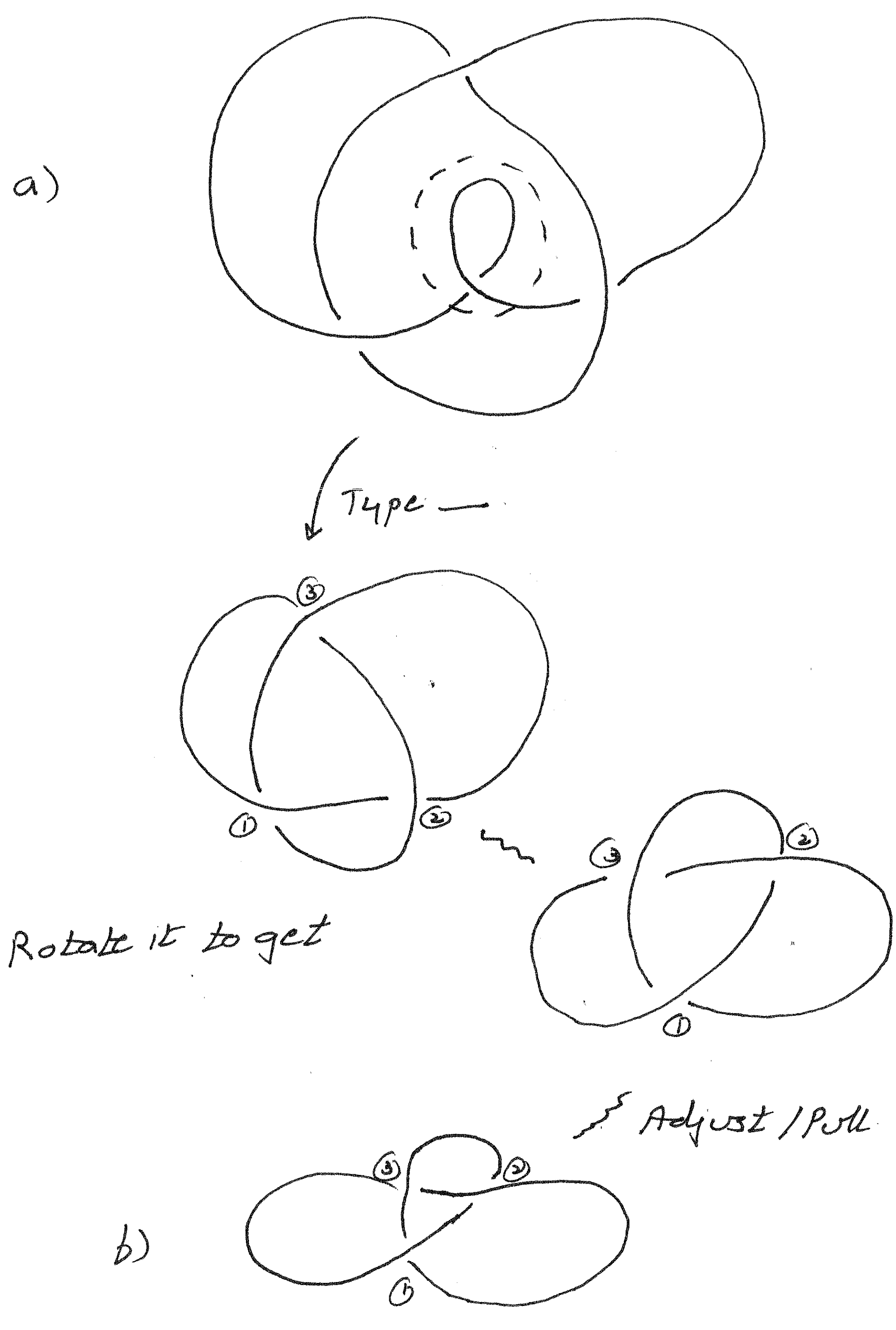
Let us go back to Q1... we know the two diagrams represent the same knot and we know how to pull the knot from Diagram a) to Diagram b). Let us see if we can make this move using only our three special Reidemeister moves...



Let us see it again for the two diagrams in Q2 which we know are the same knot so let us make Diagram a) dance into Diagram b) using our three special moves. Fill in the which Type of move they are.



Okay, lets try it one more time. We relate the two diagrams in Q4 by a sequence of Reidemeister moves. Fill in which Type of move is shown below.



It is in face a deep result in knot theory that any two knot diagrams represent the same knot if and only if they can be related by a sequence of Reidemeister moves!

3. BRUSH UP YOUR WRITHE

We've seen that it is difficult to know just by looking at two diagrams if they come from the same knot. And we've seen that two diagrams that come from the same knot are related by a sequence of our Reidemeister moves. So, one neat way of distinguishing knots would be if we could calculate a number from our diagram that did not change with Reidemeister moves.

In that case when two diagrams are related by our special moves, each move would not change this number and both diagrams would end up giving the same number. Or, in other words, if two diagrams gave us different numbers then they cannot be related by Reidemeister moves and so they must represent different knots. That would prove beyond doubt that diagrams a) and b) in Q3 and Q5 are in fact different knots (and we were in fact smart enough!)

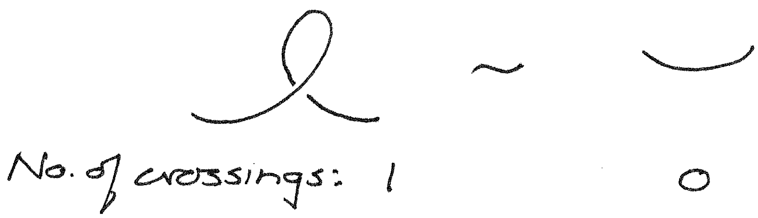
So, our aim now is to get some way of calculating a number from our diagram that does not change with Reidemeister moves. Can you come up with some way of calculating such a number from a diagram ? Think for a moment before turning the page...

What about the number of crossings ? Lets count the number of crossings in all the Diagrams a) and b) in Q1 to Q5 and record it in the table below. Fill up the empty blocks in the table below:

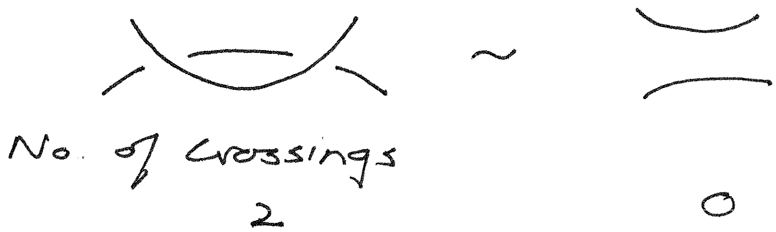
	Number of crossings in Diagram a)	**Number of crossings in Diagram b)**
Q.1	3	3
Q.2	4	
Q.3		
Q.4		
Q.5		

Well... as can be seen the Diagrams a and b in Q2 and Q4 in fact come from the same knot but they have different number of crossings... so number of crossings changes for different diagrams of the same knot. Lets see why number of crossings does not work, by looking at how it changes with our three Reidemeister moves:

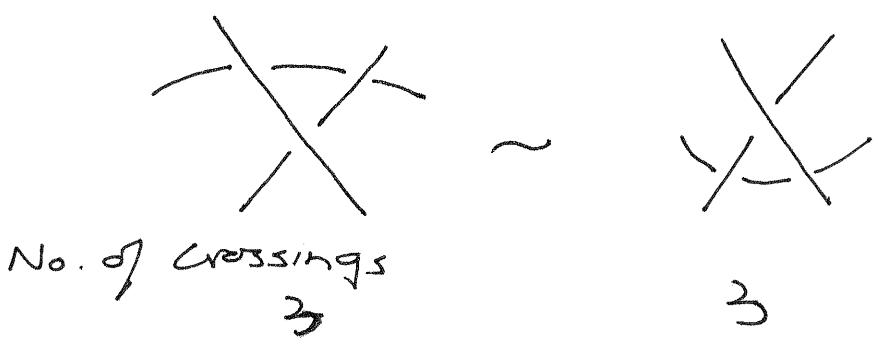
1. Move I



2. Move II



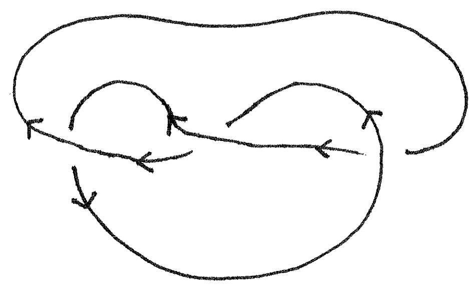
3. Move III



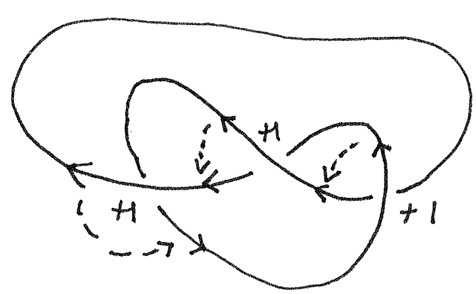
The number of crossings changes with a Move of Type I or II so clearly, the number of crossings will change as we dance from one diagram to another using our special moves. Technically, this means that the number of crossings is not an 'invariant' of the knot.

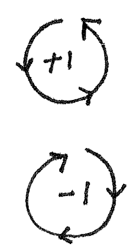
Let us try something else... how about first fixing a direction in which our knot diagram is traced (this is called, giving it an 'orientation') and then counting each crossing as a '+1' or a '-1' depending on which way the knot is drawn at the crossing. Such a count is called the writhe (pronounced as 'reeth') of the diagram. Here's how we go about calculating it for Diagram a) of Q1:

STEP 1. Give a direction to a knot by tracing it with arrows pointing in the direction of tracing. Put such an arrow on every strand coming out of each crossing(not pointing into it) as shown below.



STEP 2. Now, if you need to go in the counterclockwise direction to move the arrow on the top to the one on the bottom, then count it as +1. If you need to move in the clockwise direction count it as -1, as shown below:





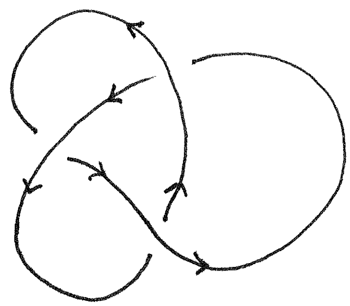
STEP 3: Add up all the 1's and (-1)'s to get the writhe of the diagram:

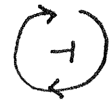
$(+1) + (+1) + (+1) = 3$

so writhe is 3.

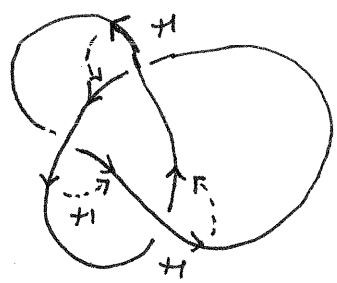
Let us calculate the writhe of Diagram b) in Q1

STEP 1. Give a direction to a knot by tracing it with arrows pointing in the direction of tracing. Put such an arrow on every strand coming out of a crossing (not pointing into it) as shown below:





STEP 2. Now, if you need to move in the counterclockwise direction to go from the arrow on the top to the one on the bottom, then count it as +1. If you need to move in the clockwise direction count it as -1, as shown below:

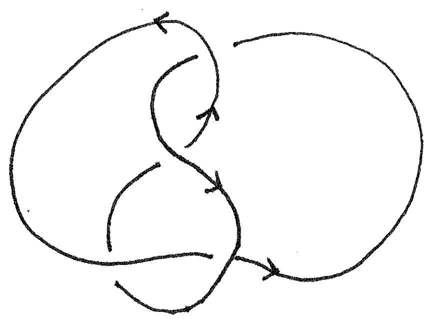


STEP 3: Add up all the 1's and (-1)'s to get the writhe of the diagram.

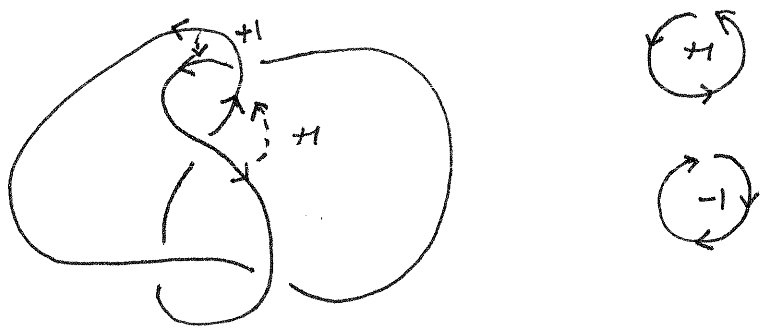
$1+1+1=3$

Now you try it... calculate the writhe of Diagram a) in Q2

STEP 1. Give a direction to a knot by tracing it with arrows pointing in the direction of tracing. Put such an arrow on every strand coming out of a crossing (not pointing into it) not already marked:



STEP 2. Now, if you need to move in the counterclockwise direction to go from the arrow on the top to the one on the bottom, then count it as +1. If you need to move in the clockwise direction count it as -1, as shown below:



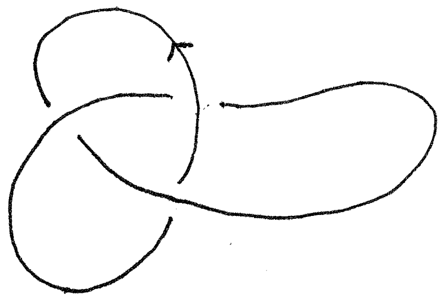
STEP 3: Add up all the 1's and (-1)'s to get the writhe of the diagram

(+1) + (+1) + ___ + ___ = ___

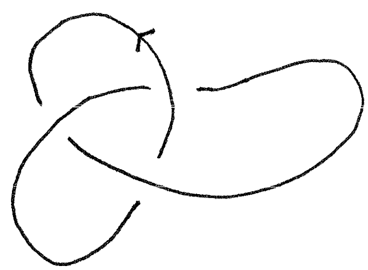
So, the writhe is ___

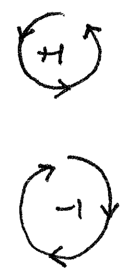
And now Diagram b) in Q2.

STEP 1. Give a direction to a knot by tracing it with arrows pointing in the direction of tracing. Put such an arrow on every strand coming out of a crossing (not pointing into it) not already marked:



STEP 2. Now, if you need to move in the counterclockwise direction to go from the arrow on the top to the one on the bottom, then count it as +1. If you need to move in the clockwise direction count it as -1, as shown below:





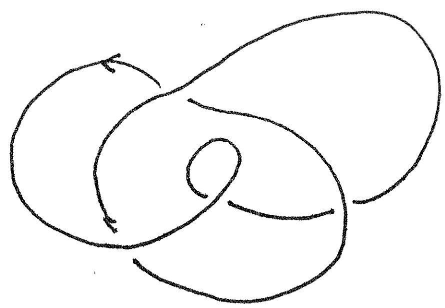
STEP 3: Add up all the 1's and (-1)'s to get the writhe of the diagram

__ + __ + __ = ____

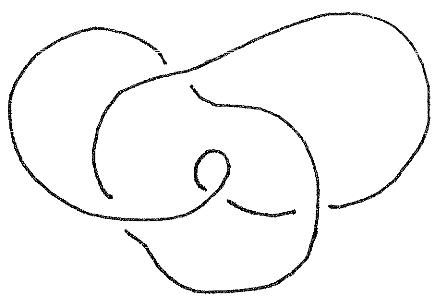
writhe is ___

Very good, now once more lets calculate it for the diagrams a) and b) in Q4. Lets start with Diagram a) of Q4:

STEP 1. Give a direction to a knot by tracing it with arrows pointing in the direction of tracing. Put such an arrow on every strand coming out of a crossing not already marked:



STEP 2. Now, if you need to move in the counterclockwise direction to go from the arrow on the top to the one on the bottom, then count it as +1. If you need to move in the clockwise direction count it as -1, as shown below:



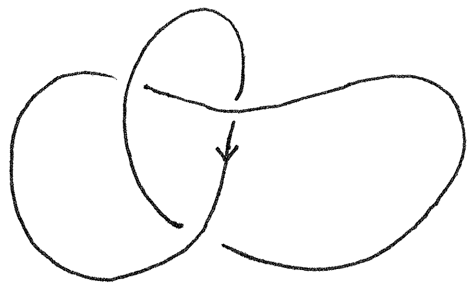
STEP 3: Add up all the 1's and (-1)'s to get the writhe of the diagram

__ + __ + __ + __ = ____

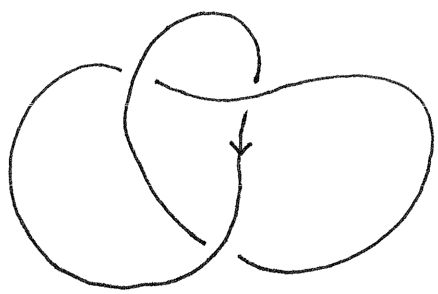
writhe is ___

And now let us finish with Diagram b) of Q4.

STEP 1. Give a direction to a knot by tracing it with arrows pointing in the direction of tracing. Put such an arrow on every strand coming out of a crossing as shown below:



STEP 2. Now, if you need to move in the counterclockwise direction to go from the arrow on the top to the one on the bottom, then count it as +1. If you need to move in the clockwise direction count it as -1, as shown below:



STEP 3: Add up all the 1's and (-1)'s to get the writhe of the diagram

___ + ___ + ____ = _____

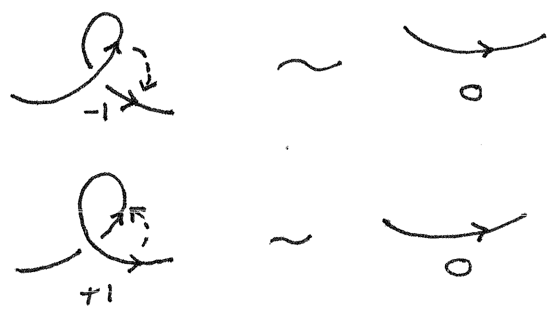
writhe is ____

Okay let us record the different writhes of the diagrams in Q1 to Q5 below (and check your answers):

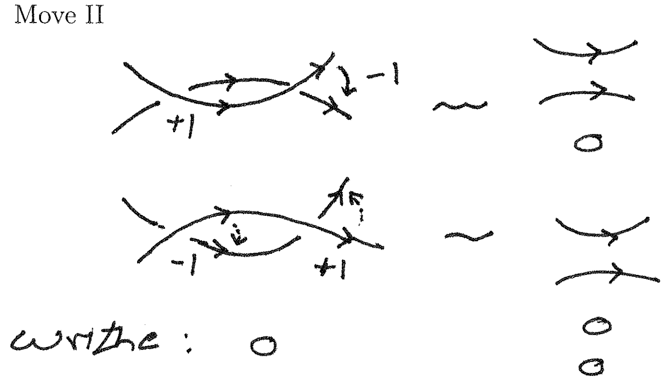
	Writhe of Diagram a)	**Writhe of Diagram b)**
Q.1	3	3
Q.2	4	3
Q.3	2	0
Q.4	-4	-3
Q.5	-5	-2

And again we see that the two diagrams in Q2 and Q4 come from the same knot but they have different writhes. Lets see why the writhe changes, by seeing how it behaves with the three Reidemeister moves:

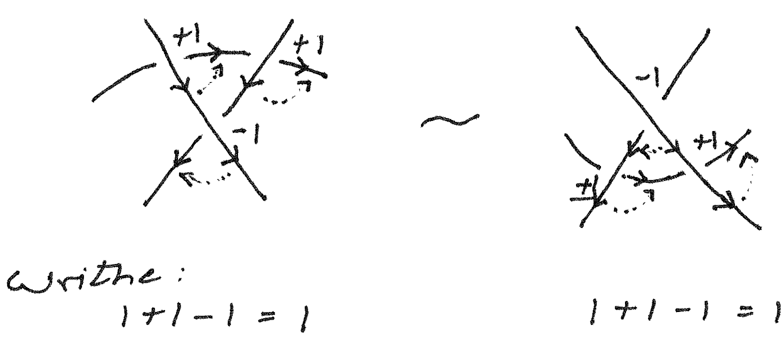
1. Move I



2. Move II



3. Move III



The writhe seems to change by $+1$ or -1 in a move of Type I and does not change in moves of Type II or III. This is an improvement over just counting the number of crossings, but we're not quite there yet... what we need is something that does not change at all with any of these moves, only then will we get a number that stays the same as we make our knot dance from one diagram to another. To do that we next calculate not a number, but a polynomial from our diagram.

4. Kauff syrup

Our aim in this section is to get a polynomial from the diagram that also does not change with Reidemeister moves of Type II or III. Unlike a usual polynomial, this one will have both positive and negative powers... such a polynomial is called a 'Laurent polynomial'. For eg

$$A^{-3}+4A^{-2}+A^{-1}+4+5A+6A^{3}+A^{5}$$

Let us get comfortable using Laurent polynomials.

Q1 Multiply $(A^{-2}-A^{2})\times(A^{3}+3A^{2}+4)$

Okay now lets calculate something called the Kauffman bracket polynomial. This is going to be the tough... but after all the work you've put in till now, this will be just one more step... or rather six more steps. The Kauffman polynomial of a diagram is calculated by the following steps. Let us calculate it for Diagram a) of Q1:

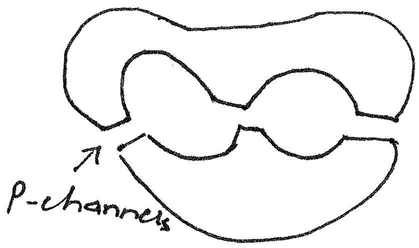
STEP 1: Number the crossings. At each crossing mark the nearby regions as P or Q as follows. Mark the region anticlockwise from the overcrossed string as P and the remaining two regions as Q:

STEP 2: A 'state' is an assignmen a P or a Q to each crossing. Write down all possible states

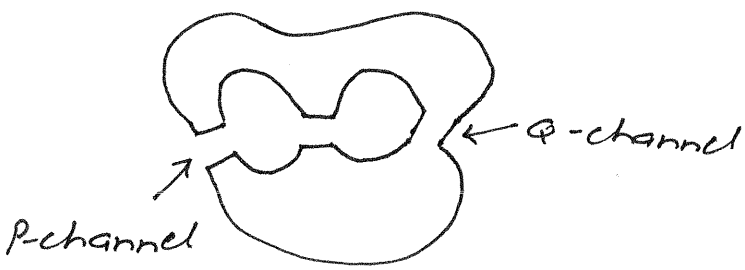
1) (P, P, P)
2) (P, P, Q)
3) (P, Q, P)
4) (Q, P, P)
5) (P, Q, Q)
6) (Q, P, Q)
7) (Q, Q, P)
8) (Q, Q, Q)

STEP 3: Remove all the crossings from the diagram and make a channel joining the Ps or the Qs as determined by the state of the crossing:

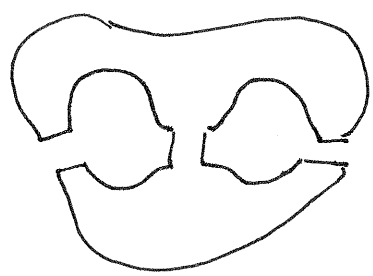
1) (P, P, P)



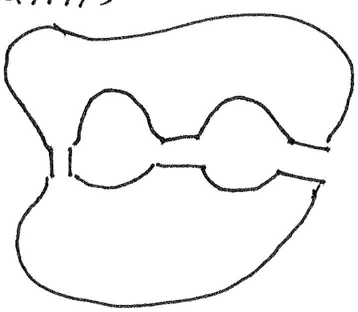
2) (P, P, Q)



3) (P, Q, P)



4) (Q, P, P)



Complete the rest.

5) (P, Q, Q)

6) (Q, P, Q)

7) (Q, Q, P)

8) (Q, Q, Q)

STEP 4: Count the number of closed curves you get after making these various channels (one for each crossing), for each state and record the following table:

States (crossing1, crossing2, crossing3)	#P-#Q	No. of closed curves(n)
(P,P,P)	3	2
(P,P,Q)	1	1
(P,Q,P)	1	1
(Q,P,P)	1	1
(P,Q,Q)	-1	2
(Q,P,Q)	-1	2
(Q,Q,Q)	-3	3

STEP 5: Get a polynomial for each state using the formula shown below:

States	#P-#Q	No. of closed curves(n)	$A^{(\#P-\#Q)}(-A^2-A^{-2})^{(n-1)}$
(P,P,P)	3	2	$A^3(-A^2-A^{-2})$
(P,P,Q)	1	1	A
(P,Q,P)	1	1	A
(Q,P,P)	1	1	A
(P,Q,Q)	-1	2	$A^{-1}(-A^2-A^{-2})$
(Q,P,Q)	-1	2	$A^{-1}(-A^2-A^{-2})$
(Q,Q,P)	-1	2	$A^{-1}(-A^2-A^{-2})$
(Q,Q,Q)	-3	3	$A^{-3}(-A^2-A^{-2})^2$

STEP 6: Add up all these polynomials and simplify to get the Kauffman bracket polynomial.

$A^3(-A^2-A^{-2})+A+A+A+A^{-1}(-A^2-A^{-2})+A^{-1}(-A^2-A^{-2})+A^{-1}(-A^2-A^{-2})+A^{-3}(-A^2-A^{-2})^2$

$\Rightarrow -A^5-A+3A-3A^{-1}(A^2+A^{-2})+A^{-3}(A^2+A^{-2})^2$

$\Rightarrow -A^5+2A-3A-3A^{-3}+A^{-3}(A^4+2+A^{-4})$

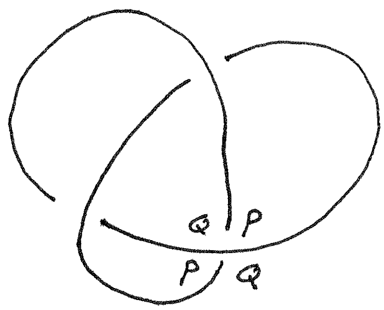
$\Rightarrow -A^5-A^{-3}+A^{-7}$

And That is our Kauffman bracket polynomial!

Yes... that was quite a handful. But let us focus on one step at a time... each step by itself is quite simple.

Let us calculate the Kauffman bracket polynomial for diagram b) of Q1.

STEP 1: Number the crossings. At each crossing mark the nearby regions as P or Q as follows. Mark the region anticlockwise from the overcrossed string as P and the remaining two regions as Q:



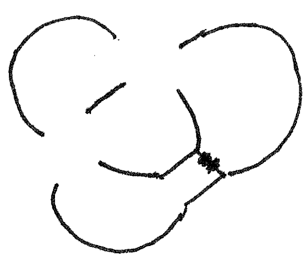
Anticlockwise

STEP 2: A 'state' is an assignmen a P or a Q to each crossing. Write down all possible states

1) (P, P, P)
2) (P, P, Q)
3) (P, Q, P)
4) (Q, P, P)
5)
6)
7)
8)

STEP 3: Remove all the crossings from the diagram and make a channel joining the Ps or the Qs as determined by the state of the crossing:

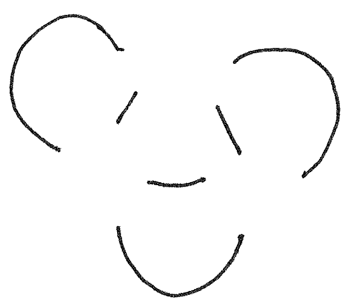
1) (P, P, P)



3) (P, Q, P)

2) (P, P, Q)

4) (Q, P, P)



Use Next Page

STEP 4: Count the number of closed curves you get after making these various channels (one for each crossing), for each state and record the following table:

Fill in the rest

States (crossing1, crossing2, crossing3)	#P-#Q	No. of closed curves(n)
(P, P, P)	3	
(P, P, Q)	2-1=1	

STEP 5: Get a polynomial for each state using the formula shown below:

States	#P-#Q	No. of closed curves(n)	$A^{(\#P-\#Q)}(-A^2-A^{-2})^{(n-1)}$

STEP 6: Add up all these polynomials and simplify to get the Kauffman bracket polynomial.

One last time... try your hand at Diagram b) of Q2.

STEP 1: Number the crossings. At each crossing mark the nearby regions as P or Q as follows. Mark the region anticlockwise from the overcrossed string as P and the remaining two regions as Q:

STEP 2: A 'state' is an assignmen a P or a Q to each crossing. Write down all possible states

STEP 3: Remove all the crossings from the diagram and make a channel joining the Ps or the Qs as determined by the state of the crossing:

STEP 4: Count the number of closed curves you get after making these various channels (one for each crossing), for each state and record the following table:

States (crossing1, crossing2, crossing3)	#P-#Q	No. of closed curves(n)

STEP 5: Get a polynomial for each state using the formula shown below:

States	#P-#Q	No. of closed curves(n)	$A^{(\#P-\#Q)}(-A^2-A^{-2})^{(n-1)}$

STEP 6: Add up all these polynomials and simplify to get the Kauffman bracket polynomial.

Okay, now that you have a hang of how this bracket polynomial is calculated, work out the polynomials for all the other diagrams after you return home.

The important thing with the Kauffman polynomial is that it does not change with Types II or III Reidemeister moves but changes by A^3 for each Type I move.

5. And finally... the Jones polynomial!

We have calculated two quantities from the diagram, the Writhe and the Kauffman bracket polynomial. Both of them do not change with Type II or III Reidemeister moves. We saw that the writhe increases by 1 every time we introduce a kink (make a Type I Move) while the Kauffman bracket gets multiplied by $(-A^3)$ each time we make this move... so why not take the polynomial $J(A) = (-A)^{-3w(D)} < D >$ where $w(D)$ is the writhe of the diagram D and $< D >$ is the Kauffman polynomial. How do you think this changes with a Type I move... think a moment before turning the page.

The polynomial does not change at all!

This is just what we were looking for... a calculation from the diagrams that does not change with Reidemesiter moves. This polynomial is called the Jones polynomial of the knot and as it does not change with our Riedemeister moves, it does not change for any two diagrams that come from the same knot.

Calculate the the Jones polynomial for each of the knot diagrams we had till now in the following table (fill in the blanks):

	Writhe $w(D)$	Kauffman bracket $< D >$	Jones Polynomial $(-A)^{-3w(D)} < D >$
Q1 a)	3	$-A^{-7} + A - A^5$	$(-A)^{-9}(A^{-7} - A^{-3} - A^5)$
Q1 b)	3		
Q2 a)	4		
Q2 b)	3		
Q3 a)	2		
Q3 b)	0		
Q4 a)	-4		
Q4 b)	-3		
Q5 a)	-5		
Q5 b)	-2		

Once we fill up this table we see that the Jones Polynomial does do the job... diagrams that come from the same knot have the same Jones Polynomial (as in Q1, Q2 and Q4). We also see that the Jones polynomials for the two diagrams in Q3 and Q5 are different... this tells us that the two diagrams in fact came from different knots. We had guessed this correctly at the start of this session but the Jones polynomial proves it beyond doubt and we have shown it is impossible to make one these diagrams dance into another... no matter how you make them dance. Mission Accomplished!

Well Done!

6. The Fineprint

(1) Technically $J(A)$ is called the Jones Polynomial only when $A = t^{\frac{-1}{4}}$. So for example if $J(A) = A^8 - A^4 + 4 - 3A^{-4}$ then we get the Jones polynomial putting $A = t^{\frac{-1}{4}}$ as $J(t) = t^{-2} - t^{-1} + 4 - 3t$.

(2) As we observed, Jones polynomial of a knot is the same for every diagram of a knot. However, we can sometimes have two different knots that produce the same Jones polynomial... so in that sense Jones polynomial doesn't completely tell you which knot you're talking about. Technically speaking, Jones polynomial is not a 'complete invariant' of the knot.

(3) There is, as yet, no algorithm for the Reidemesiter moves necessary to dance from one diagram to another. In fact, as of now, there is not even any way of saying with absolute certainty if a given knot diagram represents the unknot or not! Remember two knots can have the same Jones polynomial so if a knot gives us the Jones polynomial of an unknot that is not good enough to do the job.

(4) The website http://www.warwick.ac.uk/~maaac/TimL.html is an online calculator for Jones polynomial. You can draw your knot there and the calculator computes the Jones polynomial for you.

(5) Knot theory is a fascinating field of topology (an area of mathematics) which starts with very simple observations but still has a great many applications. So go ahead and play around with it... a good place to read more about knots is "The Knot Book: An elementary introduction to the mathematical theory of knots" by Colin Adams. Have fun...