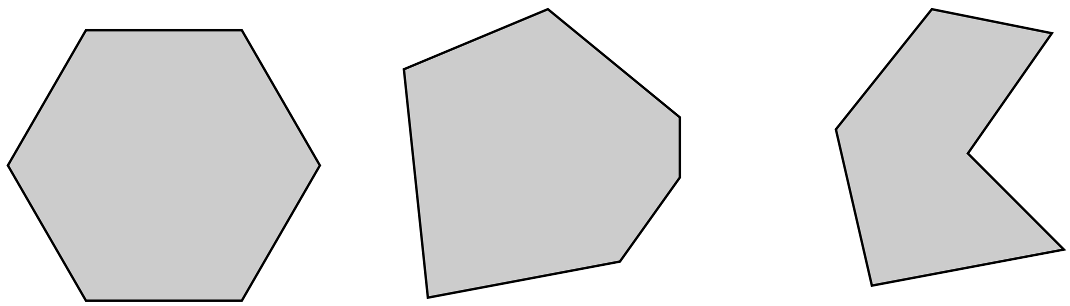
1. Convex Polygons

Definition. A shape D in the plane is *convex* if every line drawn between two points in D is entirely inside D.



Convex 6–gon   Another convex 6–gon   Not convex

Question. *Why is the third shape not convex?*

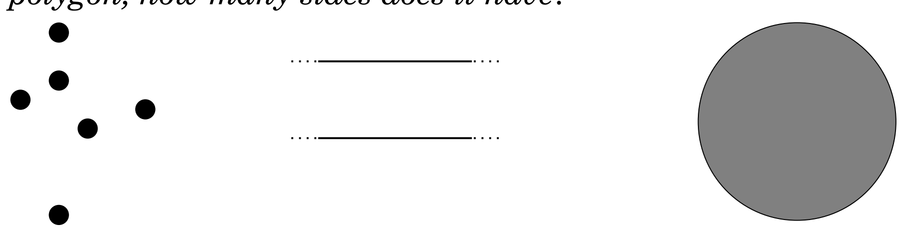
Definition. A shape D is a *polygon* if it satisfies can be described either as:

(1) The smallest convex shape enclosing a finite set of points in the plane. Or:
(2) The closed shape bounded by a finite number of lines.

A polygon with n sides is called an *n-gon*. Some have special names: a 3-gon is also called a triangle, a 4-gon a square, a 5-gon a pentagon, etc.

Question. *Give both styles of description for the two convex polygons above.*

Question. *Which of the following are convex? Which is a polygon? If a polygon, how many sides does it have?*



Smallest convex shape enclosing the above 6 points   The shape bounded by the above two lines   The shape bounded by a circle

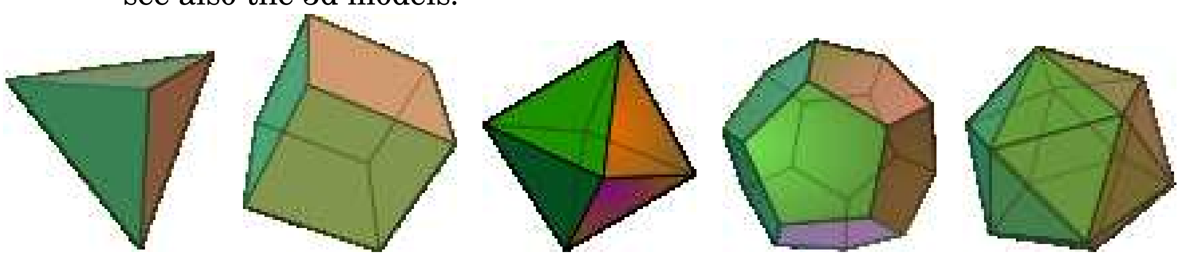
2. Convex Polyhedra

Definition. A shape is a *convex polyhedron* if it can be described either as:

(1) The smallest convex shape enclosing a finite set of points in 3 dimensions. Or:
(2) The closed shape bounded by a finite number of planes.

The first description is called the *convex hull* of the points.

Example. The platonic solids: the tetrahedron, the cube, the octahedron, the dodecahedron, and the isocahedron. Pictured below – see also the 3d models.



(Graphics source: Wikipedia)

Example. Describing the tetrahedron:

(1) Take any 4 points in 3-dimensional space that don't all lie on any plane. Their convex hull is a tetrahedron.
(2) The region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$ is a tetrahedron.

Question. *Describe the cube both as a convex hull of finitely many points and as the region bounded by planes.*

Question. *If convex polygons are 2-polytopes, and convex polyhedra are 3-polytopes, what do you think the definition of a 4-polytope should be?*

3. Vertices, Edges, and Faces

Polyhedron have vertices, edges, and faces, which I take as understood.

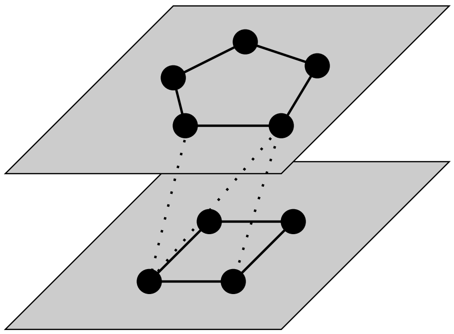
Each face of a polyhedron is a polygon.

If two polyhedra have vertices, edges, and faces that "fit together in the same way", then we regard them as being the same. For example, the cube is the same as a box, a regular tetrahedron is the same as a tall tetrahedron, etc.

Question. *Given any polyhedron, describe how to:*

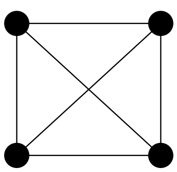
1. *Use its vertices to express it as a convex hull.*
2. *Use its faces to express it as a closed shape bounded by a finite number of planes.*

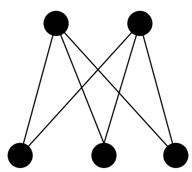
Question. *Give an example of a polyhedron which is not a platonic solid.*
Can you find a polyhedron that has both a 4*-gon and a* 5*-gon as faces?*
How about a polyhedron that has 8 *faces, but is not the octahedron?*



Did you find a polyhedron with both a square and pentagon as faces? One way to do it is as follows: put 4 points in a square in one plane, and 5 points in a pentagon in another plane. Take the convex hull of these 9 points.

A *graph* is a collection of vertices, and edges between them. They are often represented by a drawing. For example:

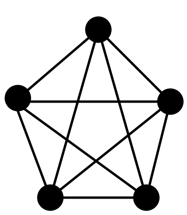




A graph is *planar* if it can be drawn in the plane without any of its edges crossing. (Such a drawing is called a *planar representation.)*

Question. *The two graphs above are both planar graphs, although the drawings given are not planar representations. Draw a planar representation of each graph.*

Question. *The* complete graph *on* 5 *vertices, pictured below, cannot be drawn in the plane. Why not?*

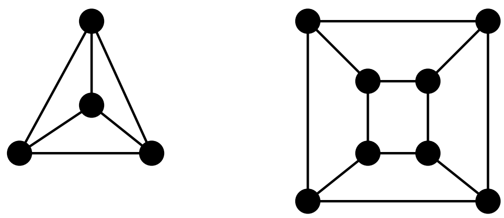


5. Graphs of Polyhedra

Definition. Any polyhedron has a graph associated with it, where the vertices of the graph are the vertices of the polyhedron, and the edges of the graph are the edges of the polyhedron. (Thus, the graph is obtained from the polyhedron by ignoring the faces.)

Since in passing to this graph you remove the "body" of the polyhedron, leaving only the "bones", it is sometimes called the *skeleton graph* or 1-*skeleton* of the polyhedron.

Example. Skeleton graphs

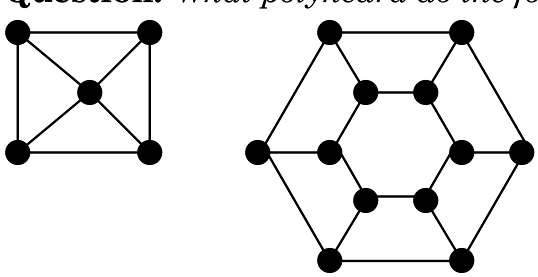


Tetrahedron   Cube

The skeleton graph of a polyhedron is always planar. To get a planar representation, think of putting your eye close enough to a face of the polyhedron that you can see the entire polyhedron. The edges will be straight, and not cross each other – draw them in the plane.

Question. *Draw a planar representation of the skeleton graph of the octahedron.*

Question. *What polyhedra do the following graphs represent?*



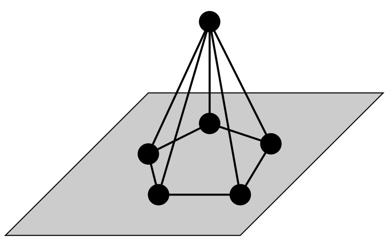
6. Lots of Examples

Unlike the situation with polygons, where every polygon is an n-gon, there are too many polyhedra to list in a simple way. But there are some interesting infinite families of polyhedra.

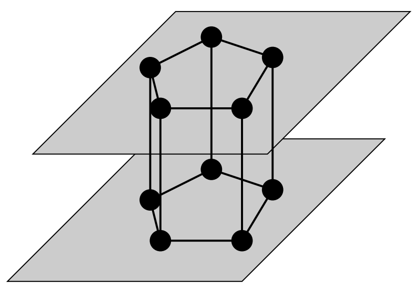
Definition. The *pyramid over the n-gon* is obtained as follows: take the n-gon in a plane, put a single point above it, and take the convex hull.

Definition. The *prism over the n-gon* is obtained as follows: take the convex hull of an n-gon together with a translated copy, as pictured.

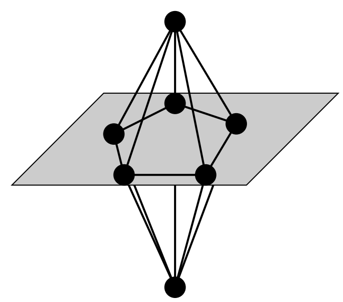
Definition. The *bipyramid over the n-gon* is obtained as follows: take the n-gon in a plane, put one point above it and another below it, and take the convex hull.



Pyramid



Prism



Bipyramid

Question. *Describe each of the tetrahedron, the cube, and the octahedron as a pyramid, prism, or bipyramid.*

Question. *What are the skeleton graphs of the pyramid over the n-gon and the prism over the n-gon?*

7. Examples and Anti-examples

Question. *Find a polyhedron which is not a pyramid, bipyramid, or prism over any n-gon.*

Question. *Find a graph which is not the skeleton graph of any polyhedron.*

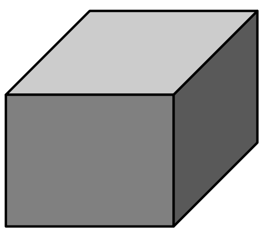
Question. *Is there any (3-dimensional) polyhedron with exactly 3 vertices? Find one, or explain why there is none.*

8. New Polyhedra From Old

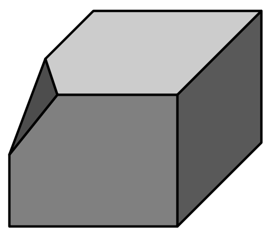
Let's mention two operations we can perform on a polyhedron to get a new polyhedron with more vertices and faces.

(1) We can "cut off" or "truncate" a vertex, by adding a new plane to the representation of the polyhedron as the region bounded by a finite number of planes.
(2) We can "cone" over a particular face, by adding a new point just over that face. (The new point needs to be close enough to maintain convexity!)

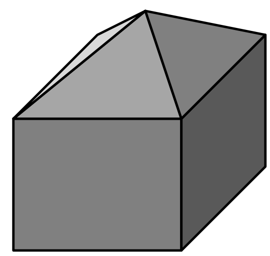
A picture is worth 1000 words:



The cube



The cube with its front upper left vertex truncated



The cube with a cone over its top face

Question. *What does coning over a face do to the skeleton graph of a polyhedron? What about truncation at a vertex? Draw some examples!*

Question. *How does coning over a face change the number of vertices, edges, and faces? What about truncation at a vertex?*

Question. *When you cone over a face, how many edges do the new faces have?*

9. Vertex-Edge-Face number constructions

We'll be interested in the relationship between the number of vertices, edges, and faces of a polyhedron.

Fact. (Euler's relation) *If a polyhedron has V vertices, E edges, and F faces, then $V - E + F = 2$.*

If you haven't seen Euler's relation before, you might be interested to check that it holds for the tetrahedron, cube, and icosahedron.

Question. *Find a polyhedron with* 5 *vertices,* 9 *edges, and* 6 *faces.*

Question. *Find a polyhedron with* 14 *vertices,* 30 *edges, and* 18 *faces.*

Question. *Find a polyhedron with* 7 *vertices,* 15 *edges, and* 10 *faces.*

Question. *Show that there is no polyhedron with* 6 *vertices,* 14 *edges, and* 10 *faces.*

10. An Edges-Faces Relation

You might have shown on the last sheet that if a polyhedron has 10 faces, it must have at least 15 edges.

From here on V, E, and F will always be the number of vertices, edges, and faces of a polyhedron.

Question. *Suppose that each face in a polyhedron is a triangle (has 3 edges). What is the relationship between E and F?*

Remark. A polyhedron where every face has 3 edges is called a *simplicial polyhedron*. A polyhedron where every vertex is in 3 edges is called a *simple polyhedron*.

Question. *Name at least two simplicial polyhedra, and at least two simple polyhedra.*

Remark. Every face of a polyhedron has at least 3 edges.

Question. *Similarly, explain why each vertex of a polyhedron is in at least 3 edges. Find a polyhedron that has a vertex contained in more than 3 edges.*

Question. *If a polyhedron has faces with any number of edges (each face having at least 3, of course!), then what is the relationship between E and F?*

11. Vertex, Edge, and Face Inequalities

Remember that V, E, and F are always the number of vertices, edges, and faces of a polyhedron.
You probably showed on Sheet 10 that $3F \leq 2E$.

Question. *Using a similar approach, show that* $3V \leq 2E$ *for any polyhedron.*

Question. *Is there a polyhedron where* $3V = 2E = 3F$*?*

Question. *By coning and/or truncating pyramids, show that if* (V, E, F) *are any triple of numbers satisfying*

$$\begin{aligned} V - E + F &= 2, \\ 3F &\leq 2E, \text{ and} \\ 3V &\leq 2E \end{aligned}$$

then there is a polyhedron with V *vertices,* E *edges, and* F *faces. (You'll have to work hard to solve this one!)*