Puzzling Probabilities

Jimin Ding

1 Multiplication Rule in Probability

1. Two cards are drawn at random without replacement from a standard deck of 52 cards.
   (a) What is the number of ways two aces can be drawn?

   (b) What is the number of ways two cards can be drawn?

   (c) What is the number of ways at least one ace can be drawn?

2. Now we randomly select 5 students from this room, what is the probability that they all have different birthdays? What is the probability that at least two will have the same birthday?

3. Suppose today we have 40 students in this room, what is the probability that at least two will have the same birthday?

4. We randomly select two students from this room, what is the probability that they have the same birthday?

2 Binomial Coefficients

5. What is the coefficient of x in the expansion of $(x+1)^2$?

6. What is the coefficient of xy in the expansion of $(x+y)^2$?

7. What is the coefficient of xy^2 in the expansion of $(x+y)^4$?

8. What is the coefficient of xy in the expansion of $(x+y)^4$?

9. What is the coefficient of x^2y^5 in the expansion of $(x+y)^7$?

10. What is the coefficient of x^2y^5 in the expansion of $(2x-3y)^7$?

11. Can you derive the coefficient of $x^r y^{n-r}$ in the expansion of $(x+y)^n$?

12. What is the coefficient of $x^2 y^2 z^3$ in the expansion of $(x+y+z)^7$?

3 Conditional Probability and Bayes' Theorem

Consider two events, A and B. Suppose we know that A has occurred. This knowledge may change the probability that A will occur. We denote by $P(B|A)$ the conditional probability of event B given that A has occurred. Bayes' theorem is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}.$$

13. One patient was told by a doctor that his medical diagnostic test of cancer was positive (which indicates the presence of cancer)! However, only 1% of the population has the this type of cancer, and we know that the sensitivity of the test is 0.95 and the specificity of the test is 0.9. So should the patient worry about his test result? What is the probability that the person actually has the cancer? (The sensitivity of a medical test is defined as the proportion of diseased people that test positive, i.e., $P(+ \mid \text{Disease})$. The specificity of a medical test is defined as the proportion of people without the disease who test negative, i.e., $P(- \mid \text{No Disease})$.)

What is the probability that the person actually does not have the disease when his test result is negative?

14. Each hereditary trait in an offspring depends on a pair of genes, one contributed by the father and the other by the mother. A gene is either recessive (denoted by a) or dominant (denoted by A). The hereditary trait is A if one gene in the pair is dominant (AA, Aa, aA) and the trait is a if both genes in the pair are recessive (aa). Suppose that the probabilities of the father carrying the pairs AA, Aa(or aA), aa are 0.25,0.5,0.25 respectively. The same probabilities hold for the mother. Assume that the matings are random and the generic contributions of the father and mother are independent.

    (a) What is the probability for a first-generation offspring carrying the pair AA?

    (b) What is the probability for a first-generation offspring carrying the pair aa?

    (c) What is the probability for a first-generation offspring carrying the pair Aa (or aA)?

    (d) What are the probabilities for a second-generation offspring carrying the pairs AA, Aa(oraA), aa?

    (e) What are the probabilities for a tenth-generation offspring carrying the pairs AA, Aa(oraA), aa? (Hardy-Weinberg law)

4 Simpson's Paradox

15. Based on Table 1, Which city have a higher mortality rate?

City	Death	Survival	Total
Sunny	1,475	98,525	100,000
Happy	1,125	98,875	100,000
Total	2,600	197,400	200,000

16. Actually above Table 1 is a summary table from the following Table 2. Based on Table 2, which city do you think have a higher mortality rate? Compare your results with the previous question, what do you see? Can you give an explanation?

	Sunny City		Happy City	
Age	Death	Total	Death	Total
<25	25	25,000	110	55,000
25-44	50	40,000	50	20,000
45-64	200	20,000	315	21,000
≥ 65	1,200	15,000	650	4,000
Total	1,475	100,000	1,125	100,000

17. A real-life example of Simpson's paradox occurred when the University of California, Berkeley was sued for bias against females who applied for graduate schools there. There were 44% of 8442 man applicants and 35% of 4321 women applicants admitted in the fall of 1973. (The difference of admission rates was so large that it was unlikely to be due to chance.) Now let us look at more closely the admission data from the six largest departments in Table 3. What is your conclusion? (Read more in: P.J. Bickel, E.A. Hammel and J.W. O'Connell (1975). "Sex Bias in Graduate Admissions: Data From Berkeley". *Science* 187 (4175): 398-404.)

	Men		Women	
Department	Applicants	Admitted	Applicants	Admitted
A	825	62%	108	82%
B	560	63%	25	68%
C	325	37%	593	34%
D	417	33%	375	35%
E	191	28%	393	24%
F	272	6%	341	7%