Finding Greatest Common Divisors

The greatest common divisor of two integers *a* and*b* is the largest positive integer *d* that divides both *a* and *b* without remainder. For example, the greatest common divisor (gcd) of 24 and 16 is 8.

1. Find gcd(42, 56) using any method.

2. An interesting way to find greatest common divisors (gcd's) is using a table. Complete the following table using integers. Where in the table can you find gcd(56, 42)?

Step	**Dividend**		**Quotient**		**Divisor**		**Remainder**
1	56	=		×	42	+	
2	42	=		×	14	+	

3. Look carefully at the numbers in the table above. Do you see a pattern? Use this pattern to complete the following table. What is gcd(13, 8)?

Step	**Dividend**		**Quotient**		**Divisor**		**Remainder**
1	13	=	1	×	8	+	5
		=		×		+	
		=		×		+	
		=		×		+	
		=		×		+	

Notice that you needed only two rows (steps) in the table to find gcd(56, 42), but five rows to find gcd(13, 8). Today you will investigate how many rows you need to find gcd's of different pairs of numbers.

You can record what you learn in the "Recording Triangle." This will help you find patterns. Read the instructions for the Recording Triangle carefully before you continue. There are some additional empty tables on the back of the Recording Triangle that you can use.

4. How many rows do you need to find gcd(4, 1) or gcd(17, 1) using the table method? Fill out the values for gcd(*a*, 1) in the Recording Triangle.

5. How many rows do you need to find gcd(4, 2) or gcd(18, 9)?

6. When *b* divides *a*, how many rows do you need? Without actually finding the gcd's, you can fill out 9 cells in the Recording Triangle column for all even numbers.

7. Fill out the entries in the Recording Triangle for the multiples of 3 in the third column, for the multiples of 4 in the fourth column, and so on.

8. The first open cell in the Recording Triangle is for $a = 3$ and $b = 2$. How many rows do you need to find gcd(3, 2) = 1 using a table? What about finding gcd(4, 3)?

9. Fill out all entries in the Recording Triangle along the diagonal for gcd(3, 2), gcd(4, 3), gcd(5, 3), …, all the way up to gcd(20, 19).

10. Use the table method to find gcd(5, 2) and gcd(7, 2). What is the pattern? Use this to complete the whole second column in the Recording Triangle.

11. You may think that a 1 in the Recording Triangle is always followed by a 2. But is that the case? How many rows do you need to find gcd(5, 3)?

12. Dividing 5 by 3 leaves a remainder of 2. Dividing 8 by 3 also leaves a reminder of 2. If you need 3 rows to find gcd(5, 3), how many rows do you need to find gcd(8, 3)? And how many rows to find gcd(18, 4)?

Think about it: 5 and 8 both have a remainder of 2 when you divide by 3. You also need the same number of rows in the table method to find gcd(5, 3) and gcd(8, 3). This is in fact a very general pattern: When *x* and *y*have the same remainder upon division by *b*, you need the same number of rows to find gcd(*x*, *b*) and gcd(*y*, *b*).

13. Use the pattern described above to complete as many entries in the Recording Triangle as possible.

14. There are more difficult patterns for you to find in the table. For example, compare the number of rows you need to find gcd(3, 2), gcd(6, 4), and gcd(12, 8). What do you notice about the rows? What do 3 / 2, 6 / 4, and 12 / 8 have in common? Now look at the entries for gcd(5, 3), gcd(10, 6), and gcd(20, 12).

15. Complete the Recording Table using the patterns that you have found. You may need to figure out some of the entries by actually computing the gcd's, but try to do that as little as possible.

We will consider one final pattern.

16. What is the first pair of numbers that takes one row? Which pair is the smallest that takes two rows?

17. Going from the upper left to the lower right, circle or color the first entry for 1, 2, 3, … in the Recording Triangle. Use your findings to fill out the following table:

Rows Needed	*a*	*b*
1		
2		
3		
4		
5		

18. Can you make a guess what will be the smallest pair that will require 6 rows in the table to find their gcd? Check your guess by actually finding the gcd for this number.

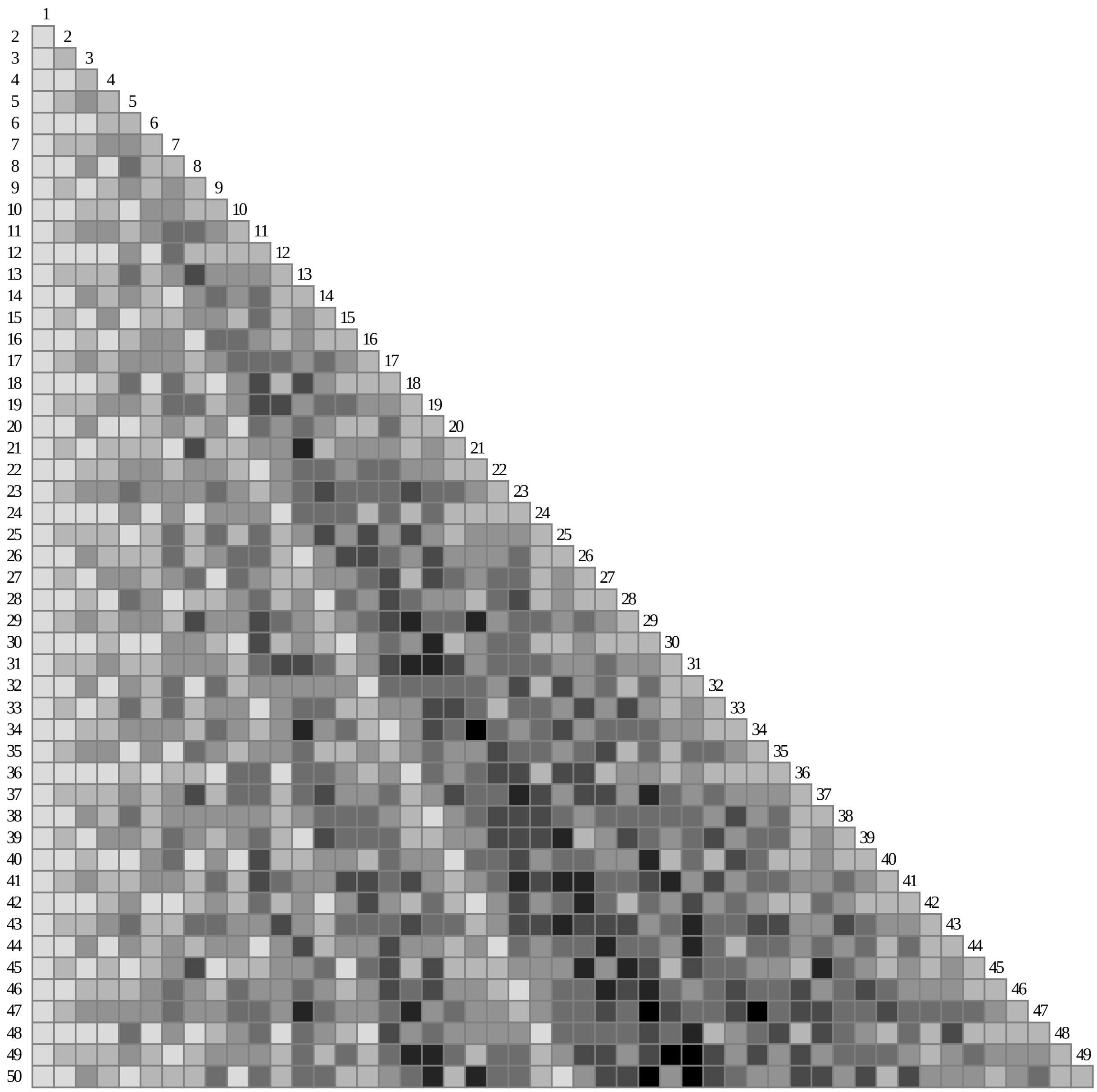
The Fibonacci numbers are the numbers 1, 1, 2, 3, 5, 8, 13, …

19. What is the next Fibonacci number? What is their pattern?

20. Consider the pair of successive Fibonacci numbers (5, 3). Where in the Recording Triangle can you find this pair? What is special about that entry?

21. What about other pairs of successive Fibonacci numbers? Can you make a guess what the connection is between Fibonacci numbers and the number of rows you need to find gcd?

22. On the next page there is a colored version of the Recording Triangle up to 50. The darker the shade of gray the more rows you need to find gcd. Find the Fibonacci numbers in the triangle.


1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50

Recording Triangle

a \ *b*	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2																			
3																			
4																			
5																			
6																			
7																			
8																			
9						2													
10																			
11																			
12																			
13																			
14																			
15																			
16																			
17																			
18																			
19																			
20																			

Instructions:

Use this triangle to record the number of rows needed to find the gcd(a, b). You look up a in the first column and b along the diagonal.

For example, you need two rows to find gcd(9, 6):

$9 = 1 \times 6 + 3$

$6 = 2\times 3$

Thus, there is a 2 in the triangle for a = 9 and b = 6.

Be careful that you record the number of rows needed and not the actual gcd!

We write a 2 in the triangle for a = 9 and b = 6, but gcd(9, 6) = 3.

Step	Dividend		Quotient		Divisor		Remainder
1		=		×		+	
2		=		×		+	
3		=		×		+	
4		=		×		+	
5		=		×		+	

Step	Dividend		Quotient		Divisor		Remainder
1		=		×		+	
2		=		×		+	
3		=		×		+	
4		=		×		+	
5		=		×		+	

Step	Dividend		Quotient		Divisor		Remainder
1		=		×		+	
2		=		×		+	
3		=		×		+	
4		=		×		+	
5		=		×		+	

Step	Dividend		Quotient		Divisor		Remainder
1		=		×		+	
2		=		×		+	
3		=		×		+	
4		=		×		+	
5		=		×		+	

Step	Dividend		Quotient		Divisor		Remainder
1		=		×		+	
2		=		×		+	
3		=		×		+	
4		=		×		+	
5		=		×		+	