Pile-subdividing Problem

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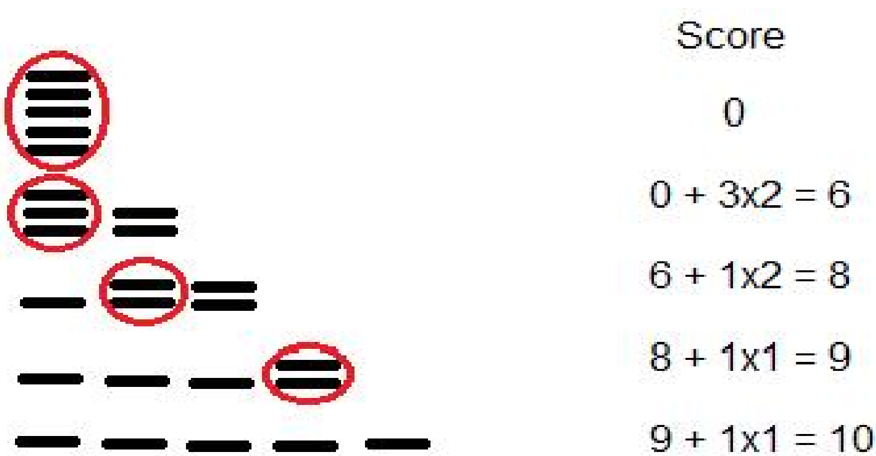
Rule of the game Suppose initially we have a single pile of n chips. At each step, we can choose one pile and split it into two new piles, and the two new piles do not need to be equal. Notice that the size of any pile should at least be 1. So eventually we will have n piles with 1 chip in each pile.

Then we attach a score to each step. For the initial pile the score is 0. Each time we divide one pile into two new piles, we add the score by the product of the size of the two new piles.

Here is an example with n=5. The circled piles are the ones we choose to split in each step. The final score we get is 10.



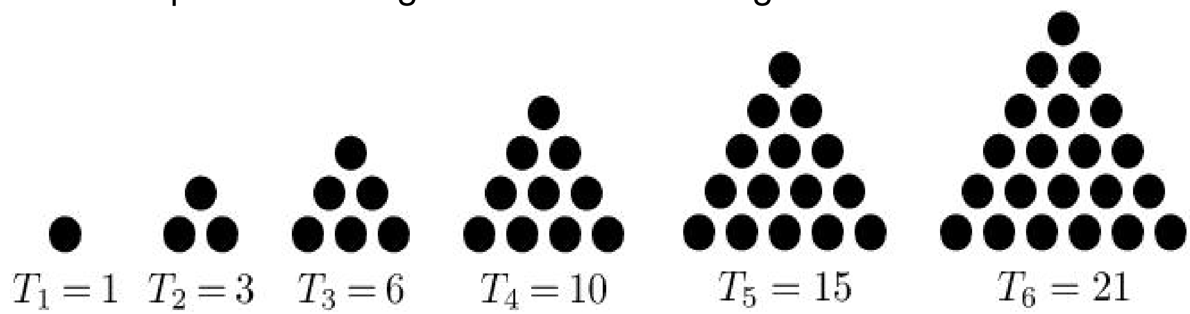
Question 1 For n=10, what is the highest score you can get? What is the lowerest score you can get? (You should try as many different subdividing strategies as possible.)

Question 2 Work on the case for n=1, n=2, n=3, n=4, n=5, n=6, n=7, n=8, n=9 and finish the table below. (Hint: when you do the case n=k, use the result you get for n $\leq$ k-1. For example, to do n=3 use result of n=1 and n=2.)

value of n	Highest score	Lowest score
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

Question 3 From the result of Question 1 and 2, what is the relationship between the score you get and the strategy you use? (Make sure your calculation in Question 1 and 2 is correct.)

Question 4 Triangular number is the number of the objects that can form an equilateral triangle. The first six triangular numbers are:



Write down the next four triangular numbers.

Question 5 Write the triangular number as a summation. (By looking at the picture, $T_1 = 1$, $T_2 = 1 + 2$, $T_3 = 1 + 2 + 3$, what is T_n?)

Question 6 What can you find about the sum of two consecutive triangular numbers? (Calculate $T_1 + T_2$, $T_2 + T_3$, $T_3 + T_4$, $T_4 + T_5$, ...) Can you give a geometric proof? (Triangular number is also the number of the objects that can form an isosceles right triangle.)

Question 7 Here is a story about the famous mathematician Carl F. Gauss. When he was a little boy, his teacher asked everyone in the class to find the sum of all the numbers from 1 to 100. To everybody's surprise, Gauss stood up with the answer 5050 immediately. The teacher asked him how it was done. Gauss explained that instead of adding all the numbers from 1 to 100, add first and last term i.e. 1 + 100 =101, then add second and second last term i.e. 2 + 99 =101 and so on. Every pair sum is 101 and there will be 50 such pairs, so 101$\times$50 = 5050 is the answer.

Use the result of Question 5 and follow Gauss's method to get a formula for T_n.

Question 8 Let S_n denote the score you get with the initial pile size n. Can you give a formula for S_n? (Compare the table you get in Question 2 with the triangular number you get in Question 4. Also use the result of Question 7.)

Question 9 Check the validity of the formula in Question 8 by calculating the score you get when n=11, 12, 13, 14, 15. (Use the result of Question 3.)

Question 10 From Question 9 you know the formula in Question 8 is correct for n$\leq$15. Now suppose the formula is correct for n$\leq$k (and we do not know whether it is correct for n=k+1). How can we determine whether the formula is correct for n=k+1? Just follow the three steps below.

First, find the relationship among S_{k+1}, S_{k+1-m} and S_m, where m is a positive integer with m$\leq$k-1. (Recall the rule of the game.)

Second, get a formula for S_{k+1} by the relationship you get.

Last, compare the formula you get for S_{k+1} with the formula in Question 8 with n=k+1.

Note Mathematical induction is a technique to prove a statement asserted about every natual number.

The principle of mathematical induction: if
(1) the statement is true for n=1,
(2) when the statement is true for n$\leq$k, the statement is true for n=k+1,
then the statement is true.

Actually, you have already showed with initial pile size n the score you can get is always $\frac{n(n-1)}{2}$, no matter what subdividing strategy you use.