# Highway Planning for Mathematicians

**Washington University Math Circle February 19, 2012**

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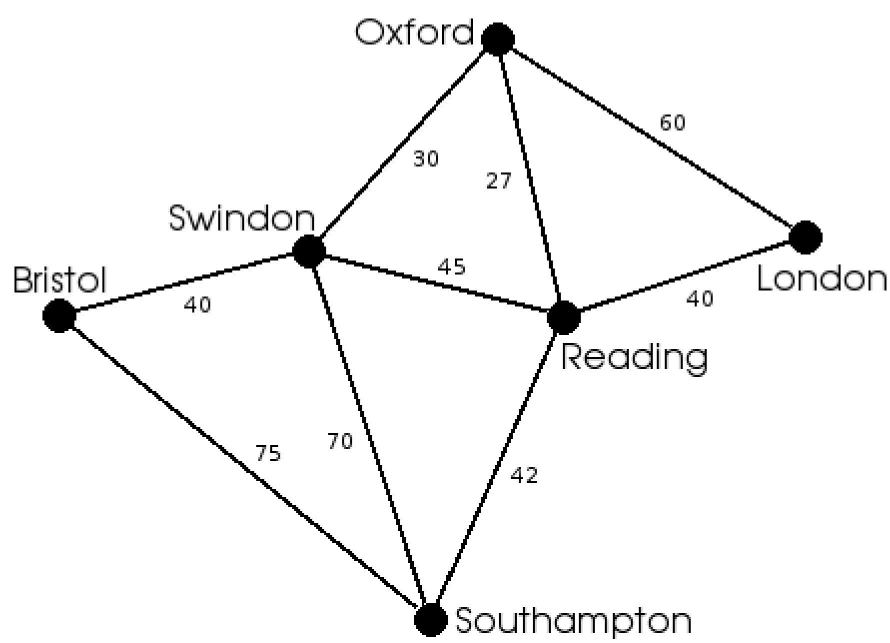
Suppose you are given the location of several cities and asked to connect them with a network of highways. Which network would require the least total miles of road? Actually, the most popular solution would be to just run a highway directly between *every* pair of cities—people are willing to pay for more roads if they can get where they want to go faster—but it is a good way to think about minimal networks, which are studied extensively by research mathematicians to model circuits and communications networks.

## Warm-up Activity: Minimal Spanning Trees

For each possible route the mileage is given. Which roads should you choose to get the shortest total mileage?




Total mileage of your network:_______

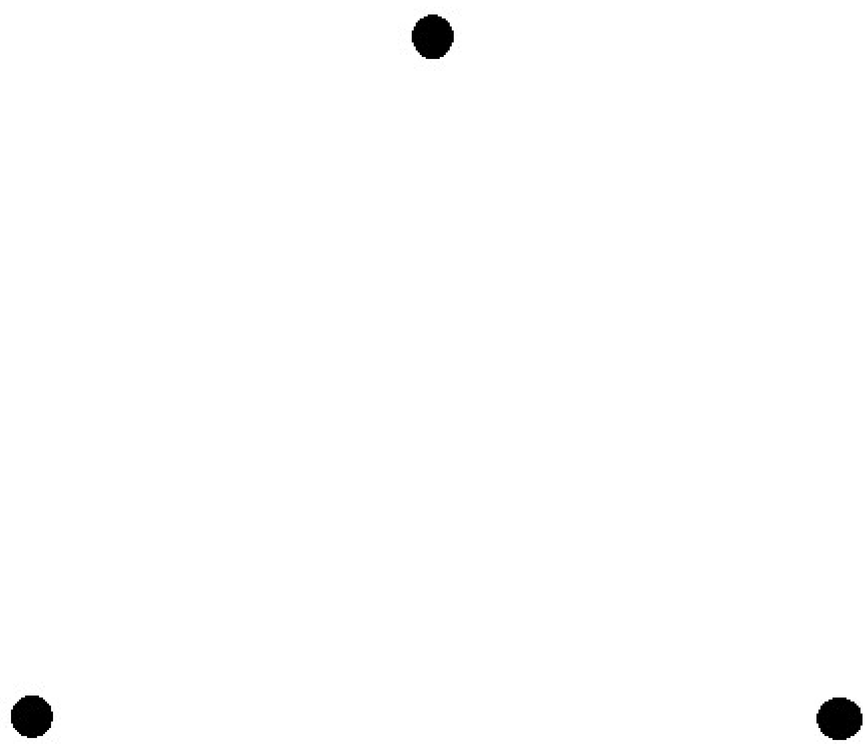
*Think about:* Can you recall the mathematical meaning of a "tree"? How is it related to shortest network problems?

*Note: from here on your will need your ruler!*

# Activity 1: The Steiner Problem I

# (The Equilateral Triangle)

Given three points, what is the shortest length of a network connecting all the points?



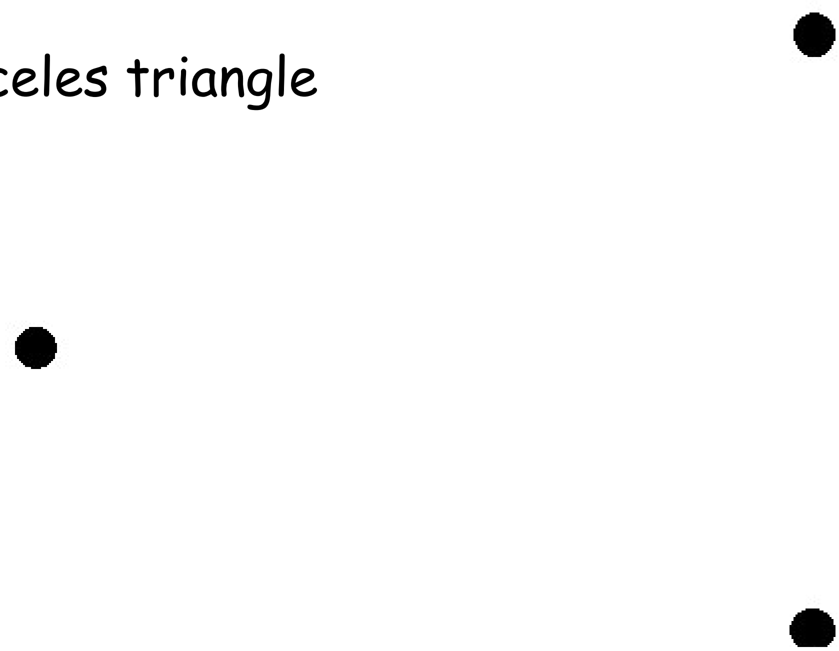
Total Length:___________

*Think about:* If we scale the length of the three sides to one inch, can we calculate the length without measuring? (This requires some trigonometry —if you haven't had any don't worry about it!)

# Activity 2: The Steiner Problem II

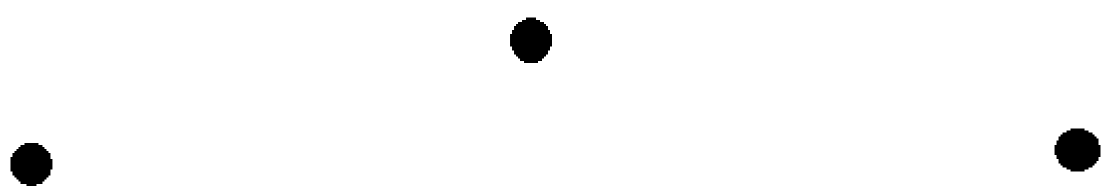
## (Other Triangles)

- An isosceles triangle



Total Length:___________

- A very long triangle



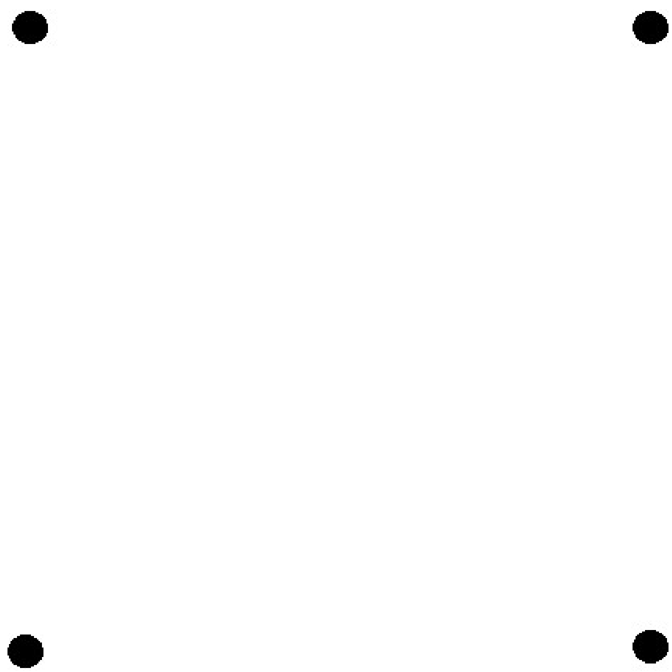
Total Length:__________

***Think about:*** What can we say about the added "Steiner point"? Is it always a good idea to add one? If we do use one what can we say about where to put it? How many edges will it connect?

# Activity 3: The Steiner Problem III (Quadrilaterals)

Given four points, what is the shortest possible network connecting all the points?

- The square



Total Length:__________

- A rectangle

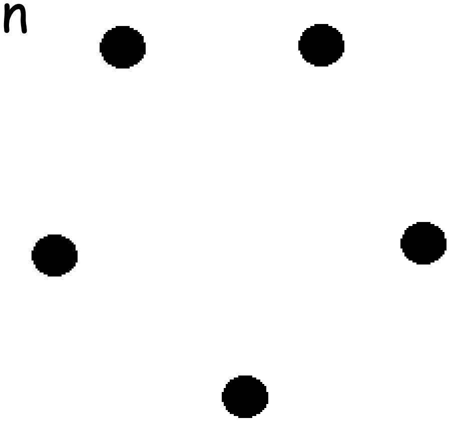
Total Length:__________

Think about: Can you use the idea from the triangle solution, or is there a better alternative?

## Activity 4: The Steiner Problem IV (Other Polygons)

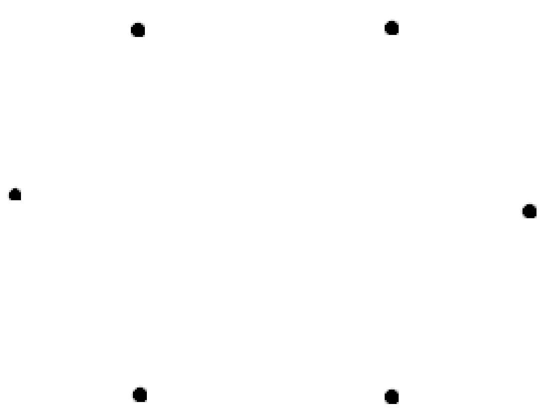
Given five or more points, what is the shortest network connecting all the points?

- The regular pentagon

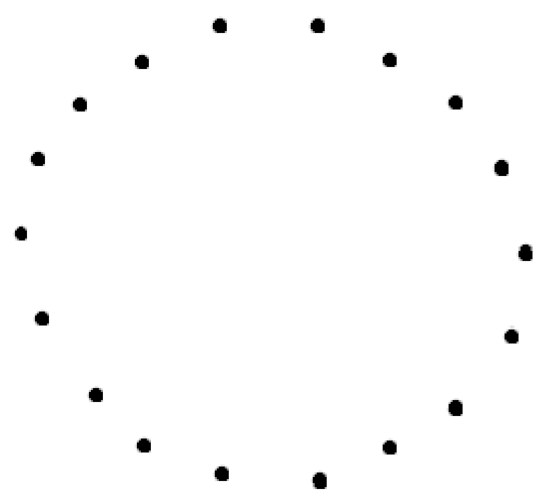


Total Length:__________

The regular hexagon



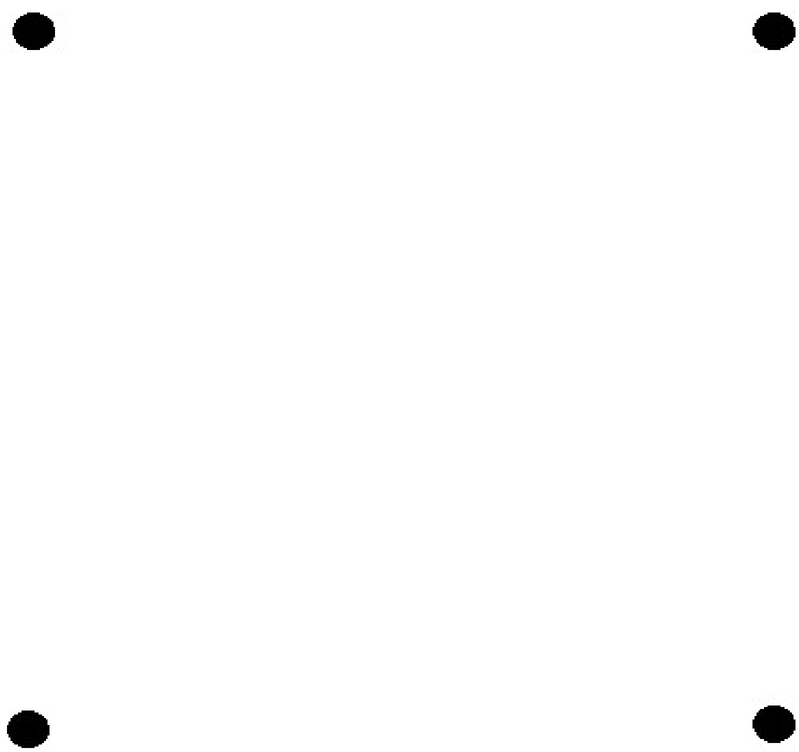
The regular octadecagon



Think about: For the last two examples, finding the total length starts to become complicated. Can you think of a general rule that might hold for polygons with a large number of vertices?

## Bonus Question 1: The Opaque Square

For this problem imagine the four points are the four corners of your yard and the network is made of fences. What is the least amount of fencing you can use so that *no one can see through your yard*?

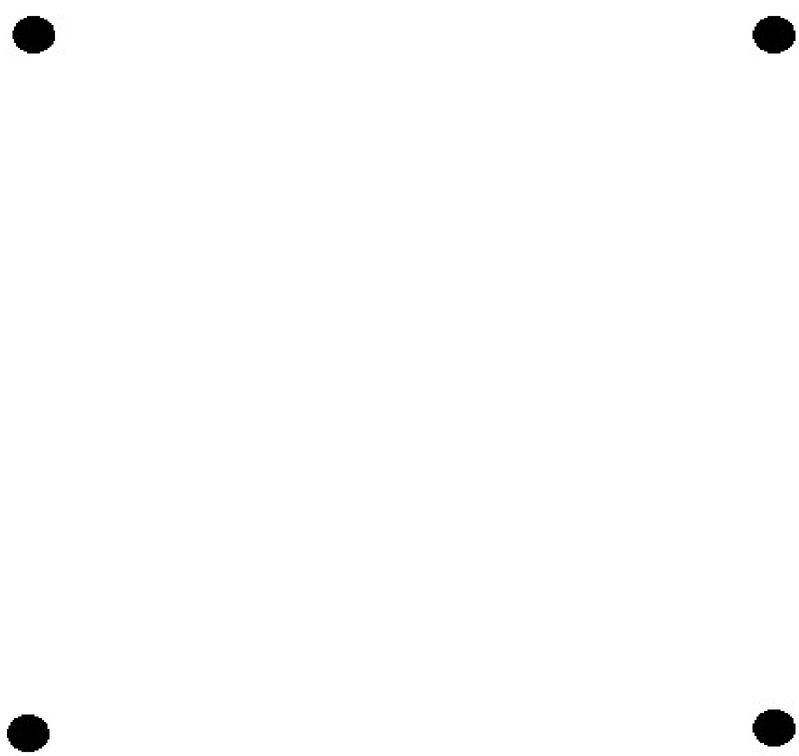


Think about: Which ideas from the Steiner problem still apply? Which do not apply?

This is a wide open question, but have fun and see if you can think of different approaches!

Bonus Question 2: Added cost for Steiner Points

Suppose we compute the cost of the road network by adding the length of the roads PLUS a cost for each Steiner point you need. What network is the cheapest?



Total Cost: Length in cm _____ + number of Steiner points _____ = ________