A Mathematician Guards The Art Gallery

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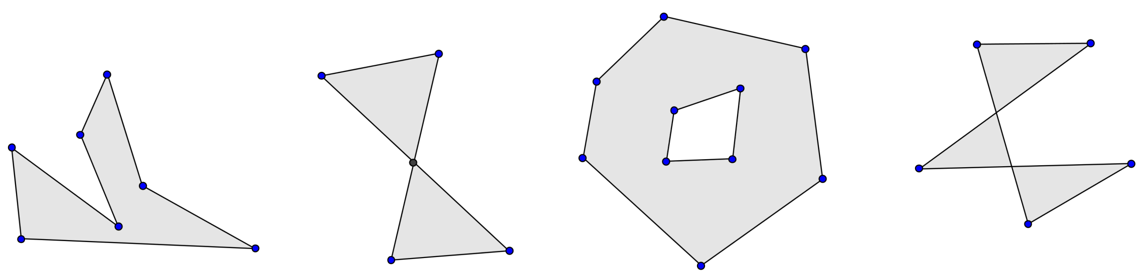
1 Polygons, Diagonals, and Triangulations

Definition 1. *A **polygon** P is a closed region bounded by a collection of line segments (called edges) that does not cross itself. The points where edges meet are called vertices. The vertices and edges are called the **boundary** of the polygon which we write as ∂P.*

For the following problems we insist on a few things:

1. We do not want to consider polygons whose boundaries cross themselves, i.e. we want our polygons to be *simply connected*.
2. We insist that vertices happen at "bends" in the curve and do not occur in the middle of an edge. In other words, we do not want three points on the same edge, i.e. no three points collinear on an edge.
3. We do not want any holes in our polygon.

Question 2. *Which of the following shapes are polygons? Why or why not?*



Definition 3. *A polygon P is called **convex** if given any two points x, y inside the polygon, the line $\overline{xy}$ is contained entirely inside of P. Otherwise, the polygon is called **nonconvex** or **concave**.*

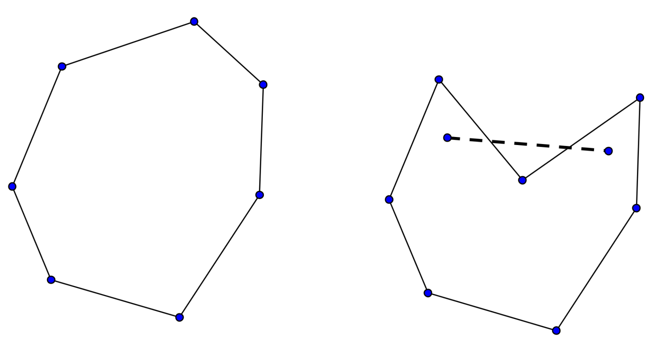


Figure 1: A convex polygon (left) and a nonconvex polygon (right)

I suggest you experiment with both convex and nonconvex polygons when working on the following problems, in fact nonconvex polygons tend to be the most interesting!

Definition 4. *A **diagonal** of a polygon is a line segment connecting two vertices of P and lying completely inside of P.*

Question 5. *What is the smallest number of vertices required to make a polygon?*

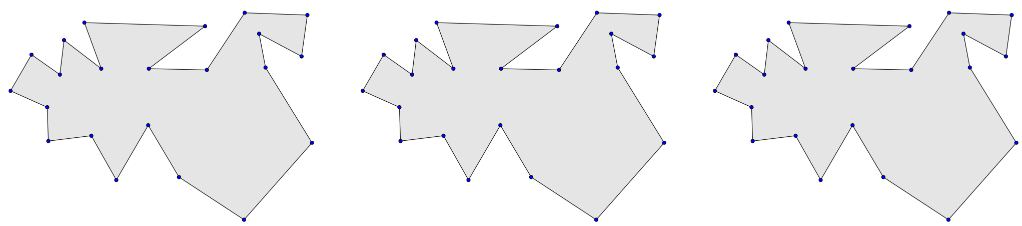
Question 6. *Draw several polygons and experiment with drawing diagonals between vertices.*

Question 7. *Does every polygon have a diagonal? Can you come up with an explanation to back up your claim? (Think about what you saw in Questions* 5 *&* 6*).*

Question 8. *Create several polygons and draw as many noncrossing diagonals as you can. What happens to each polygon? Does this happen to every polygon?*

Definition 9. *A* ***triangulation*** *of a polygon P is a decomposition (or breaking down) of P into triangles by a set of noncrossing diagonals. We continue adding diagonals until we cannot add anymore.*

Question 10. *Draw different triangulations for the polygon below by connecting different vertices together. Try to make them as different as possible!*



Question 11. *How many diagonals does each polygon in question 10 have?*

Question 12. *How many triangles does each polygon in question 10 break down into?*

Question 13. *Draw some new polygons on your own with different numbers of vertices and answer the previous two problems relating to your new polygons.*

Question 14. *For any triangulation of a polygon with n vertices (where $n = \{3, 4, 5, 6, \ldots\}$), can you conjecture(or guess) a relationship between the number of vertices and the number of diagonals in terms of n? What about the relationship between the number of vertices and the number of triangles in each triangulation in terms of n? Do you think this is true for all polygons?*

Question 15 (Challenge)**.** *Consider a* ***convex*** *polygon. How many* ***different*** *triangulations does each polygon with $3, 4, 5, \ldots, n$ vertices have?*

1.1 Optional Problems

The answer to this problem is known for every polygon with n vertices (where $n > 3$) and brings rise to the **Catalan Numbers**, which describe (among other things) the number of triangulations of a convex polygon. Formally, the Catalan number C_n is given by

$$C_n = \left(\frac{(2n)!}{(n+1)!n!}\right)$$

and C_n gives the number of triangulations of a convex polygon with $n + 2$ vertices! The notation $n!$ is best explained with an example. $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. In other words, we just multiply the number 5 by every number below it! The first few Catalan Numbers for $n > 3$ are $2, 5, 14, 42, 132, 429, 1430, 4862, \ldots$

Question 16 (Challenge)**.** *How many different triangulations does the **nonconvex** polygon in question 10 have? (Hint: A lot!)*

The answer to this problem for **nonconvex** polygons is not so simple. The best we can do is get an upper and lower bound for the number of triangulations of a nonconvex polygon.

Question 17. *Can you create a polygon with only one triangulation? How about creating a polygon with only two different triangulations?*

Question 18. *Can you conjecture the lower bound for triangulations? What about the upper bound? (Hint: Look to previous problems!)*

Question 19 (Super Challenge!)**.** *What about creating a polygon with only three different triangulations? This is part of an interesting problem that I was presented this year and do not know the answer to. Can **you** construct such an polygon?*

For nonconvex polygons with $n+2$ vertices, the best we know is that the number of triangulations of a nonconvex polygon is somewhere between 1 and C_n. For the polygon in question 10 (which has 22 vertices), the number of possible triangulations is somewhere between 1 and $C_{20} = 24,466,267,020$!

2 Guarding Art Galleries

Definition 20. *We say a point x inside a polygon P **sees** or **guards** a point y inside P if the line segment $\overline{xy}$ does not touch the outside of the polygon. In other words, the line segment $\overline{xy}$ is contained inside the polygon or touches ∂P.*

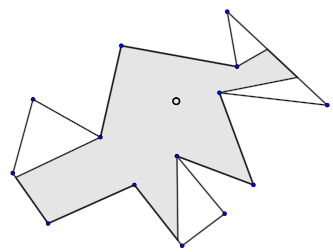
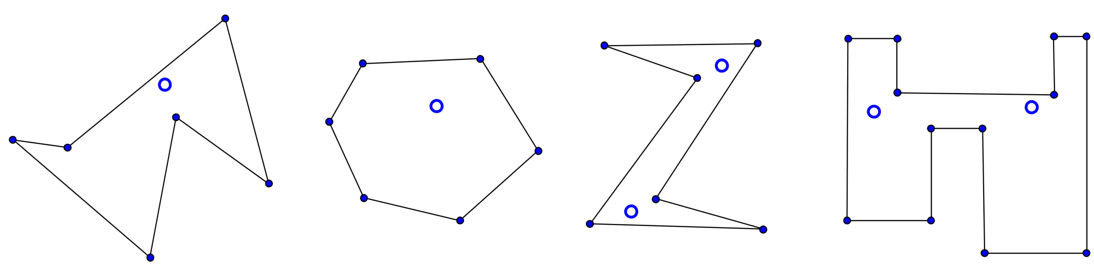


Figure 2: The shaded area represents the part of the polygon the guard can see.

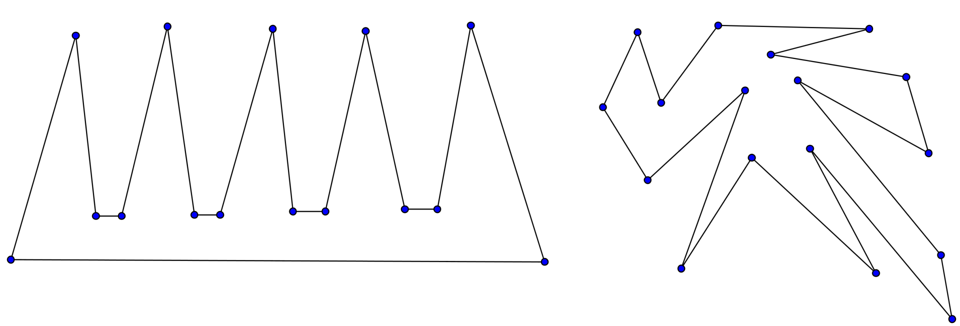
2.1 The Art Gallery Problem

We want to consider the inside of P as art galleries, and we want to place guards (which are represented by points) inside P such that every point of the interior of P can be seen by at least one guard. We will explore this problem, as well as variations on the problem.

Question 21. *For the following polygons, determine whether the entire gallery is covered, and if not show where the guards cannot see.*



Question 22. *For the following polygons, determine the minimum number of guards needed to cover the art gallery. Try experimenting with drawing your own polygons!*



This problem has a known answer. The minimum number of guards needed to guard a polygon with n vertices is $\lfloor \frac{n}{3} \rfloor$ (Note: this notation represents the floor function, which says that we find $\frac{n}{3}$ and then round down to the nearest number). This holds even if we say the guards have to stand at the vertices of the polygon.

Question 23. *Does every polygon need $\lfloor \frac{n}{3} \rfloor$ guards? Can you give an example of polygons that do not need $\lfloor \frac{n}{3} \rfloor$ guards? Are they necessarily convex?*

Question 24. *On the flip side, can you find polygons which MUST have $\lfloor \frac{n}{3} \rfloor$ guards?*

We will end with some of my favorite variations on this problems, and point out which ones have solutions (as far as I know).

1. (Fortress Variant) How many guards must be placed outside a polygon to cover the exterior of polygons? This is known to require $\lceil \frac{n}{2} \rceil$ for some polygons, but some polygons can be covered with less (this is the ceiling function, which is similar to the floor function except we round up to the nearest number).
2. (Edge Guards) If every edge is considered to be a fluorescent light bulb, how many edges need to be lit to light up the entire interior of the polygon? This is conjectured to be $\lfloor \frac{n}{4} \rfloor$, but has not been proven so.
3. (Mirrors) If the edges of a polygon are perfect mirrors, is one guard enough to see everything? To my knowledge, this is still an open problem.
4. (Wifi Variant) Suppose we replace guards with wireless internet transmitters. These wireless signals have the unique property that they may travel through a finite number of walls before they stop. We can then ask the following question: If we assume the wireless signal can penetrate two walls before ending, how many transmitters are needed to cover the entire polygon? This is not known to my knowledge, but a very interesting variant I was presented recently!

You can make your own variants too, and see if you can find answers for them! By altering small parts of a mathematical problem, we can ask many other related questions. This is one way in which mathematicians find new problems to work on...we modify existing problems! What variants can you come up with?