The Fibonacci Sequence

Fibonacci (a.k.a. Leonardo of Pisa) asked the following problem in 1228.

A man put one pair of rabbits in a certain place entirely surrounded by a wall. How many pairs of rabbits can be produced in a year, if the nature of these rabbits is such that every month each pair bears a new pair, a male and a female, which from their second month on, becomes productive?

There are some implicit assumptions:
i) Rabbits never die or become infertile
ii) The first pair is newly born

Question 1. What is the answer to Fibonacci's problem?

The Fibonacci sequence is defined this way.

$$\begin{aligned} F_1 &= 1 \\ F_2 &= 1 \\ F_n &= F_{n-1} + F_{n-2}, \qquad \text{for } n = 3, 4, 5, \ldots . \end{aligned}$$

Question 2. Calculate the first 12 Fibonacci numbers.

How many different ways can you climb a flight of 3 stairs, if you can go up either one stair or 2 stairs with each step?

Answer:

$$3 = 1 + 1 + 1 = 2 + 1 = 1 + 2,$$

so there are three ways.

Question 3: What if there were 5 stairs?

Question 4: What if there were 6 stairs?

Question 5: What if there were 11 stairs?

Question 6: What if there were n stairs? Why?

Choose any two positive whole numbers, J_1 and J_2 say (keep them small to make the arithmetic manageable).

Make your own Fibonacci sequence by inductively defining

$$J_n = J_{n-1} + J_{n-2}, \qquad \text{for } n = 3, 4, 5, \ldots.$$

Question 7: What are the first 10 elements in your sequence?

Question 8. Add up the first 10 elements. Divide by 11. What do you notice?

Question 9: Do you think what you found depends on what two numbers you started with? Why?

Question 10. For $n = 2, 3, 4, \ldots$, calculate the ratio $\dfrac{F_n}{F_{n-1}}$. What do you notice?

Let us define a new sequence by the formula

$$L_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

Question 11: What is L_1?

Question 12: What is L_2?

Question 13: What are L_3, L_4 and L_5?

Question 14: What would you guess L_{12} is?

Question 15. It can be proved that $L_n = F_n$ for every n. What do you think the strategy would be to prove this?

$1+\sqrt{5} = 3.236...$ and $1-\sqrt{5} = -1.236....$
So when n is large, $(1+\sqrt{5})^n$ is much bigger than $|(1-\sqrt{5})^n|$.
So a good approximation for L_n is

$$A_n \;=\; \frac{(1+\sqrt{5})^n}{2^n\sqrt{5}}.$$

Question 16: What is $\dfrac{A_n}{A_{n-1}}$? How does this relate to Question 10?

Homework (voluntary)

Fibonacci numbers crop up frequently in nature. In particular, for many flowers the number of petals is a Fibonacci number. See *e.g.*

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html#plants

for pictures of flowers and seed-heads, and a description of their relation to Fibonacci numbers.

Exercise: Draw your own flower this way.

1. Put a dot in the center of a page. This is the center of the flower.
2. Choose some angle x.
3. Put a dot for your first petal a distance 1 away from the center.
4. Rotate by x, and put your second dot a distance $\sqrt{2}$ away.
5. Rotate by x, and put the third dot a distance $\sqrt{3}$ away.
6. Continue like this — the n^{th} dot will be rotated x from the previous one, and be a distance $\sqrt{n}$ from the center.

If x is 1/8 of a revolution — 45^o — then the pattern is crowded at the center, and spread out far away.

If x is chosen as a Fibonacci ratio of revolutions — *e.g.* $\dfrac{21}{13}$ revolutions which is 582^o which is the same as 222^o — the pattern looks much more like a real daisy.