CUBES, CUBES, CUBES!
WASHINGTON UNIVERSITY MATH CIRCLE

ADAM G. WEYHAUPT

These questions can be thought about in any order, so find a problem you think is interesting!

A. **Warm up: Painting a cube.**

Paint the outside of a $10 \times 10 \times 10$ cube yellow. How many cubes were painted?

B. **Eating a cube.**

A $3 \times 3 \times 3$ cube of cheese is built from small $1 \times 1 \times 1$ cheese cubes. A mouse eats the $3 \times 3 \times 3$ cube by beginning with one of the eight $1 \times 1 \times 1$ corner cubes and proceeding always to an adjacent cube. (Adjacent means that the two small cubes share a face.) Can the mouse eat in such a pattern so that she finishes her meal on the centermost cube?

Date: November 16, 2014.

C. Building colored cubes.

Imagine you have 8 plain wooden cubes. Can you paint these cubes red and blue in such a way that they can be assembled into a $2 \times 2 \times 2$ cube whose outside is all blue or all red? (You should be able to build a blue cube and a red cube.)

Imagine you have 27 plain wooden cubes. Can you paint these cubes red and blue and green in such a way that they can be assembled into a $3 \times 3 \times 3$ cube whose outside is all red, blue, or green? (You should be able to build all three of these cubes.)

D. Coloring a cube.

You want to paint a cube so that each of the six faces is a solid color, but so that no two adjacent faces have the same color. What is the fewest number of colors required for this?

If you are given 4 colors, how many possible different cubes can you make? Two cubes should be considered the "same" if they are the same after moving them around.

E. Summing the cubes. Our goal in this problem is to find a formula for the sum of cubes and to *prove* that the formula always works!

What is $1^3 + 2^3$?

What is $1^3 + 2^3 + 3^3$?

What is $1^3 + 2^3 + 3^3 + 4^3$?

What is $1^3 + 2^3 + 3^3 + 4^3 + 5^3$?

It might be very hard to find a pattern in these numbers. Let's try some geometry. You need to know that

$$\text{the area of a triangle is } \frac{1}{2}(\text{base}) \times (\text{height}).$$

You also need to know that there is an easy way to calculate

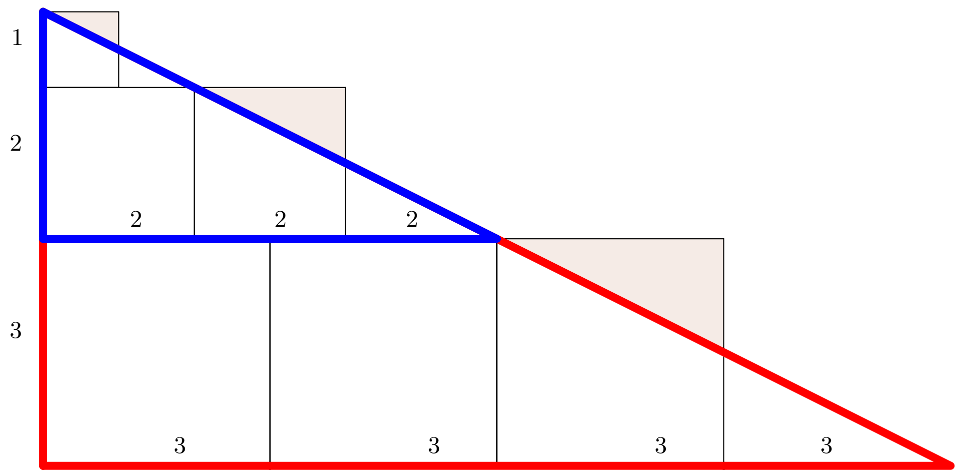
$$1 + 2 + 3 + \cdots + 100.$$

In fact,

$$1 + 2 + 3 + \cdots + 100 = \frac{100 \times 101}{2}.$$

In general,

$$1 + 2 + 3 + \cdots + n = \frac{n \times (n+1)}{2}.$$



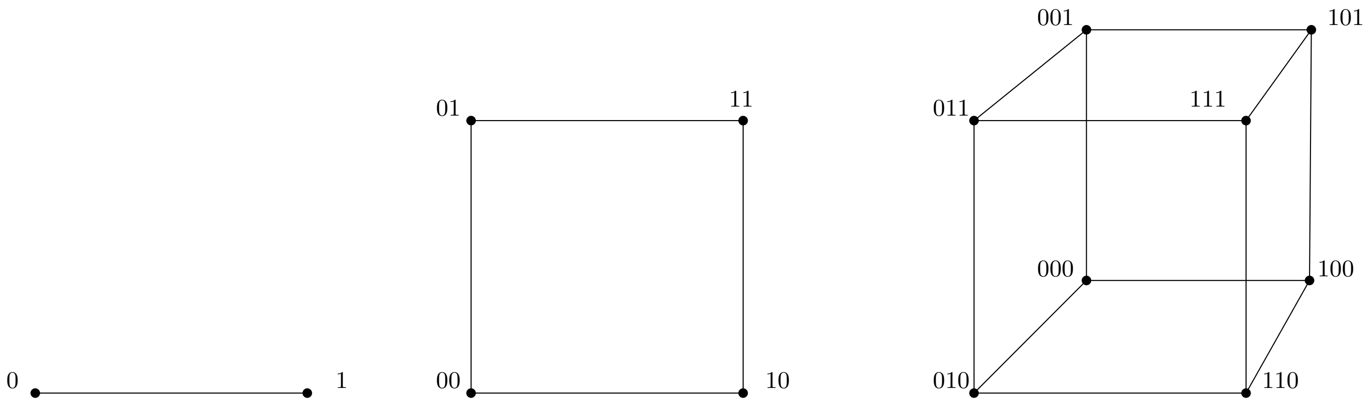
What is the area of the big BLUE triangle?

What is the area of the big RED triangle?

Using these as a guide, draw a triangle that has area $1^3 + 2^3 + 3^3 + 4^3$.

Using these as a guide, draw a triangle that has area $1^3 + 2^3 + 3^3 + \dots n^3$? Using your picture, write a *formula* for the area of the triangle. Test your formula on $1^3 + 2^3 + 3^3 + 4^3 + 5^3$ to see if it is correct.

F. **Higher dimensional cubes.** We've been dealing with cubes in three dimensional space, but cubes exist in every dimension.



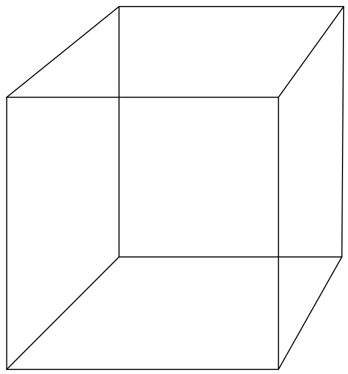
A four dimensional cube is called a "hypercube".

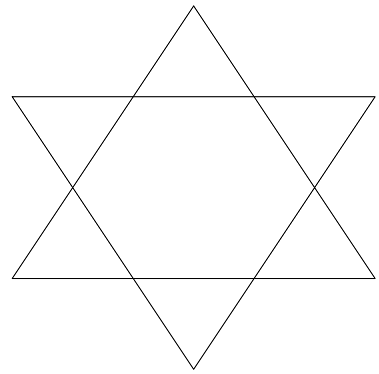
> How many vertices do you think a four dimensional cube has?

> I've used a labeling system on the vertices of the cubes that has two vertices connected if the labels have only one number changed. A two-dimensional cube (square) has "faces" that are one-dimensional squares. A three-dimensional cube (cube) has "faces" that are two-dimensional cubes (squares). A four-dimensional cube (hypercube) has "faces" that are ...?

> I've used a labeling system on the vertices of the cubes that has two vertices connected if the labels have only one number changed. Using this system, can you *draw* a hypercube? (Or build one?)

G. Graphs from cubes. There is an unusual way to represent a cube using a *graph*.

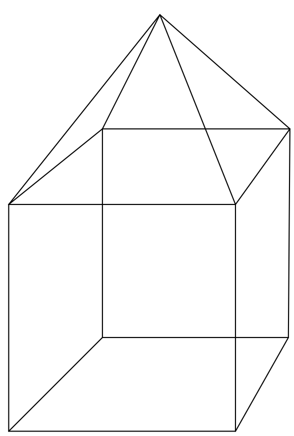




This diagram represents the types of connections on the cube. Each face of the cube is an edge of the graph. (There are six edges in the graph and six faces on the cube.) Two edges intersect in the graph exactly when two faces are adjacent.

Color edges of the graph and faces of the cube so that corresponding edges match with corresponding faces.

What does a vertex (point) in the graph represent?



Consider a house with a roof on it. Can you draw the corresponding graph?

Stick 3 cubes together along faces in different ways. Can you draw the corresponding graph?

Build other shapes by sticking cubes together along faces. Can you always draw the corresponding graph using straight lines? Or might you sometimes get stuck?

Southern Illinois University Edwardsville, Department of Mathematics and Statistics, Edwardsville, IL 62026

E-mail address: aweyhau@siue.edu