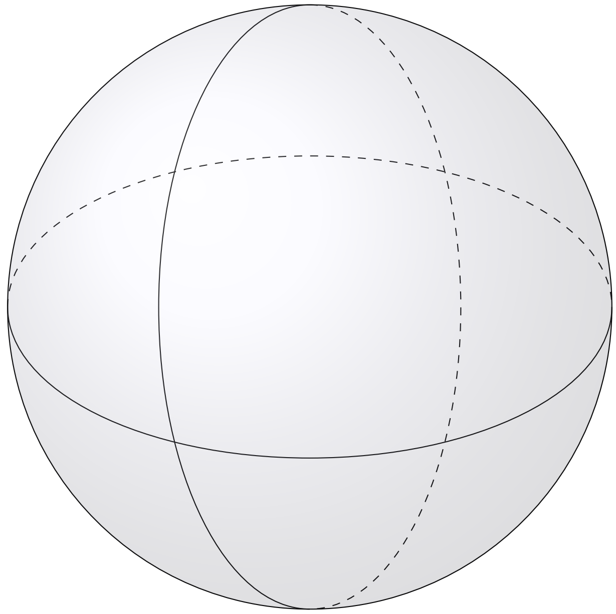
WUSTL Math Circle Sept 27, 2015

The K-12 geometry, as we know it, is based on the postulates in Euclid's *Elements*, which we take for granted in everyday life. Here are a few examples:

1. The distance between two points is given by the length of a unique straight line.
2. The sum of the angles in a triangle is $180°$.
3. π, the ratio between the circumference of a circle and its diameter is constant.
4. Through a point not on a line l, there is precisely one line parallel to the line l.

Today, this is known as Euclidean geometry. For over two millenniums, Euclidean geometry, or the geometry of flat space, stood unchanged. However, in the early 19th century, other forms of geometry begin to emerge, where geometric objects in non-flat spaces are studied.

Question 1. What is the shortest distance between two points on a sphere?



Question 2. On a sphere, is the shortest distance between two points represented by a UNIQUE curve?

On a curved surface like a sphere, the shortest curve connecting two points are called the geodesics. The straight lines on a flat surface are replaced by the geodesics on curved surfaces.

Question 3. What are the geodesiccs on a sphere?

A triangle on a sphere is then three geodesic segments that meet at three vertices.

Question 4. Is it true that the sum of the angles of a triangle on a sphere is still $180°$?

Question 5.

- Find a triangle on the sphere such that the sum of the angles is $200°$;
- Find a triangle on the sphere such that the sum of the angles is $270°$;
- Find a triangle on the sphere such that the sum of the angles is $340°$;

Question 6. What happens when a triangle is getting smaller?

Let us take two great circles on a sphere that pass through the north and the south pole, the two geodesic circles are “parallel” at the equator, but they grow closer as they approach the poles.

This means that “parallel” geodesics on a curved space do not stay “parallel” forever.

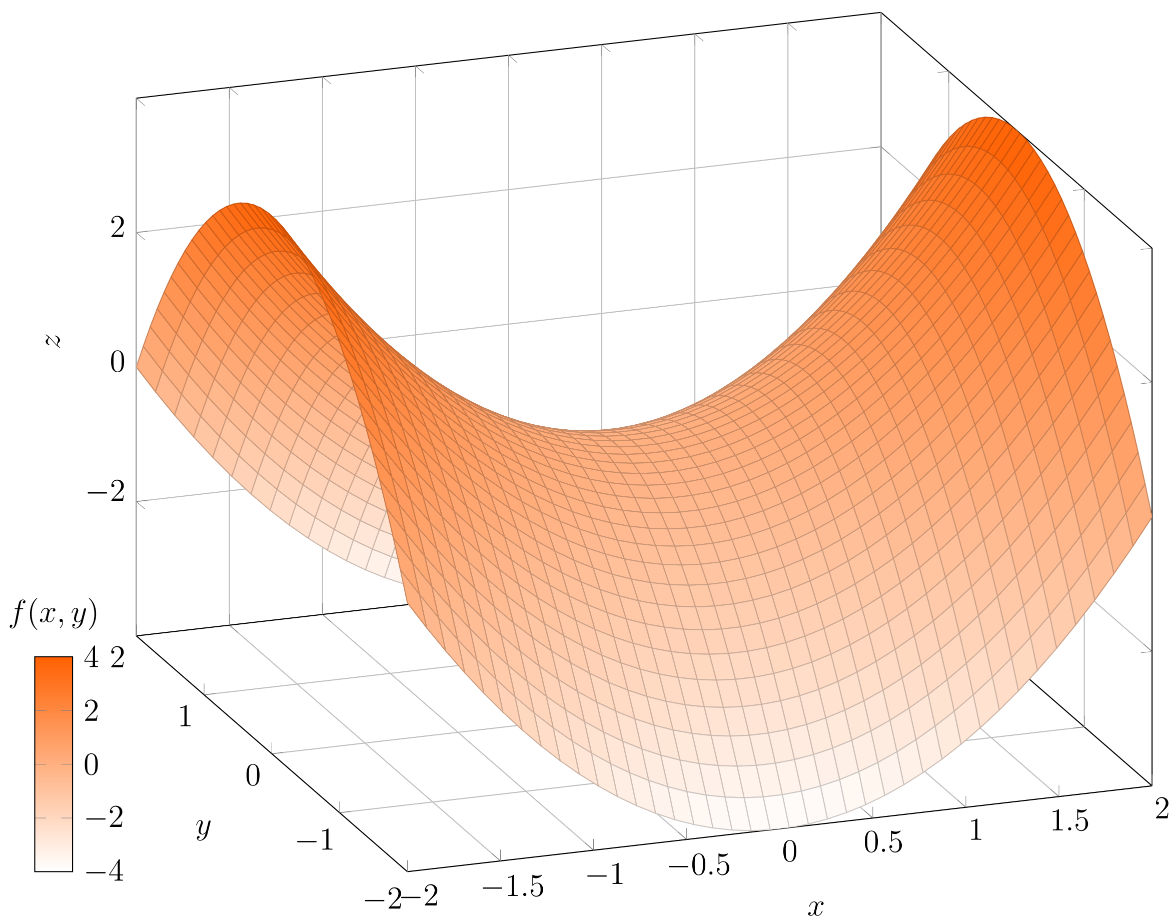
On a curved surface, if two “parallel” geodesics always grow closer, then we say that the surface is parabolic. For example, the sphere is parabolic.

On the other hand, if two “parallel” geodesics always stay at the same distance forever, then we say the surface is flat.

Finally, if two “parallel” geodesics grow apart, then we say that the surface is hyperbolic. For example, the sphere is parabolic.

What kind of surfaces are hyperbolic? Here is an example, the hyperbolic paraboloid given by

$$z = x^2 - y^2.$$



Question 7. Draw a few geodesics to verify that the hyperbolic paraboloid is indeed a hyperbolic surface. That is, the geodesics grow apart.

We now introduce two very important examples of hyperbolic surface. Both these examples are proposed by Italian mathematician Eugenio Beltrami, and named after French mathematician Henri Poincare. The first is the so-called Poincare half-plane.

For two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the upper half-plane, i.e. $y_1, y_2 > 0$, there is a unique semi-circle passing through P_1 and P_2 that ends on the x-axis. This semi-circle is the geodesic passing through P_1 and P_2. The two points on the boundary are called ideal points. The distance between P_1 and P_2 is defined to be

$$d((x_1, y_1), (x_2, y_2)) = \text{arcosh}\left(1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2y_1 y_2}\right)$$

where

$$\text{arcosh}x = \ln\left(x + \sqrt{x^2 - 1}\right), \qquad x \geq 1.$$

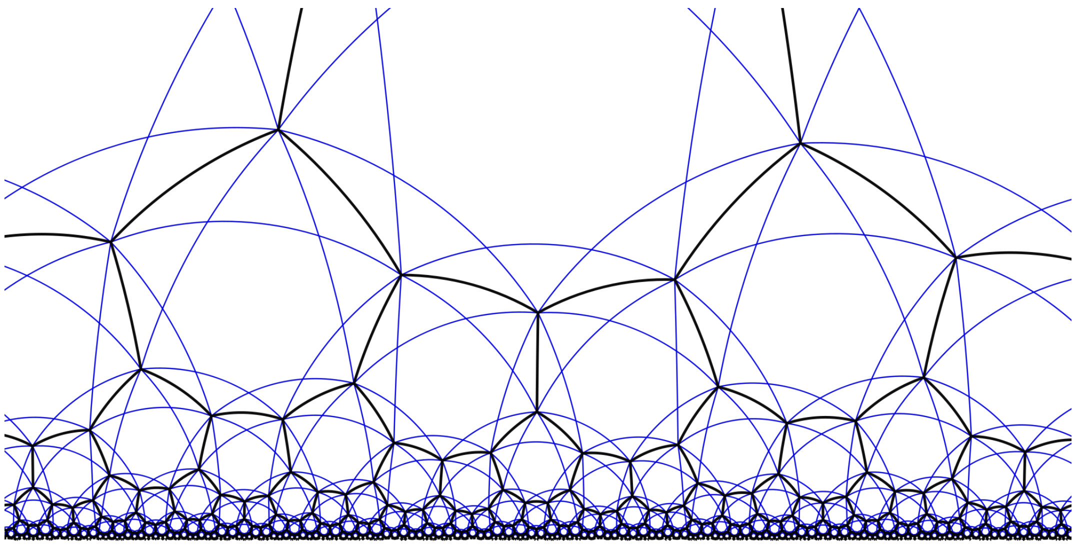


Figure 1: All the heptagon are identical

Question 8. Visually verify that the geodesics on the Poincare half-plane grow apart.

Apart from the semi-circle geodesics, there is also a special set of geodesics on the Poincare half-plane, namely the vertical lines starting on the x-axis.

Question 9. Calculate the distances:

- $d((0, 10), (1, 10))$;
- $d((0, 2), (1, 2))$;
- $d((0, 1), (1, 1))$.

How do you interpret the result?

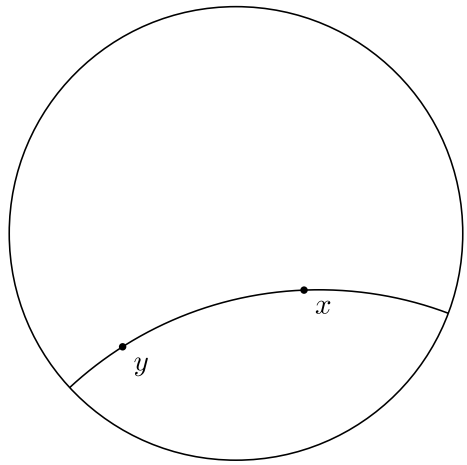
Question 10. Verify that if $x_1 = x_2 = x$, then we have

$$d((x, y_1), (x, y_2)) = |\ln y_2 - \ln y_1|.$$

The second example is the Poincare disk.

For two points x and y inside the disk, there is a unique circle passing through x and y and intersecting the boundary circle at right angles. This circle is the geodesic passing through x and y. The two points on the boundary are called ideal points. Label the points in order as a, x, y and b. The distance between x and y is defined to be

$$d(x, y) = \ln \frac{|ay| \cdot |xb|}{|ax| \cdot |yb|}.$$



Question 11. Visually verify that the Poincare disk is a hyperbolic surface. That is, the geodesics grow apart.

We conclude with a few artworks of the Poincare disk.

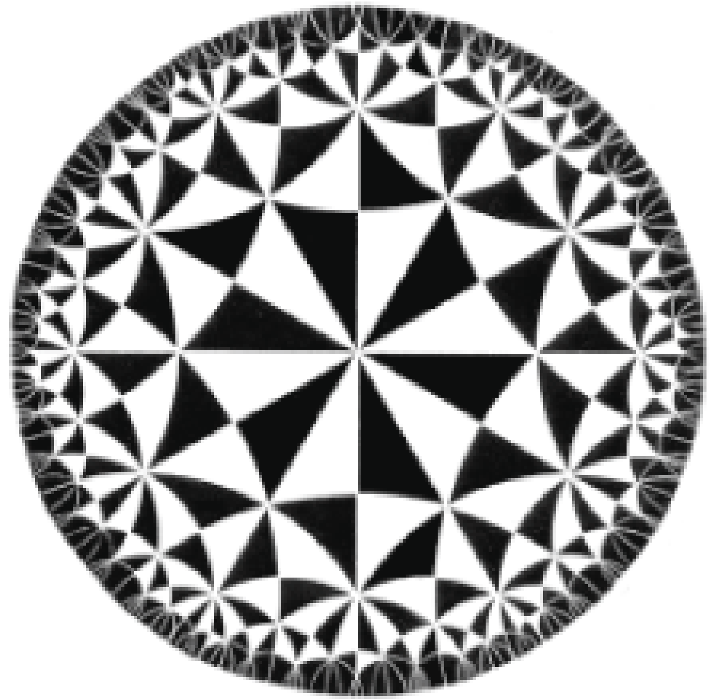


Figure 2: All the triangles are idential.

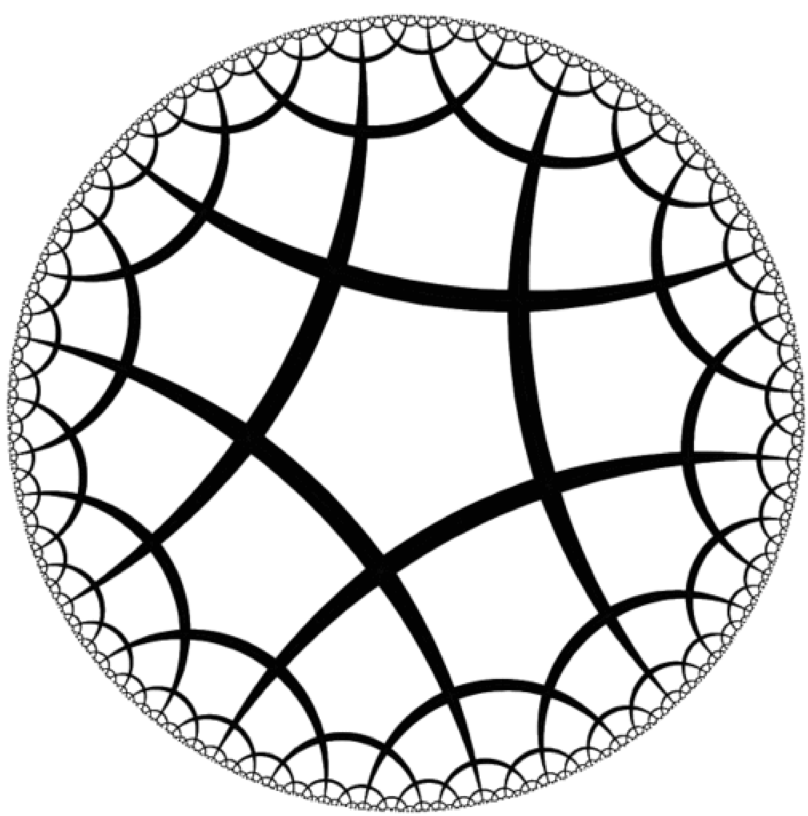


Figure 3: All the pentagons are idential.

Figure 4: Circle Limit III, by M. C. Escher 1959

Answer 1. On a sphere of radius r, the shortest distance is represented by the shorter segment of the cricle of radius r passing through the two points. There circles are called the great circles.

Answer 2. No. For example between the north pole and the south pole, there are infinitely many great circles connecting them.

Answer 3. There are exactly the great circles.

Answer 4. No. For example, we may take the north pole and two points on the equator. The triangle with these three vertices has the property that each of its three angles is 90°, so the sum is 270°. In fact, the sum of the angles of a traingle on the sphere is always bigger than 180°.

Answer 6. As a triangle on the sphere gets smaller, the surface also gets more flat. Thus, the sum of its angles approaches 180°.

Answer 7. Eh... It is too difficult to draw them on a computer. Ask me in person. :-)

Answer 8. Figure 1 should tell. :-)

Answer 9. Please use your calcualtor. You should find that

$$d((0,10),(1,10)) < d((0,2),(1,2)) < d((0,1),(1,1)).$$

This means that the vertical geodesics on the Poincare half-plane grow apart as they approach the x-axis.

Answer 10. If $x_1 = x_2$, then we have

$$1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2y_1 y_2} = \frac{y_1^2 + y_2^2}{2y_1 y_2}.$$

Let us assume $y_2 \geq y_1$, then we have

$$\begin{aligned} d((x,y_1),(x,y_2)) &= \ln\left(\frac{y_1^2 + y_2^2}{2y_1 y_2} + \sqrt{\left(\frac{y_1^2 + y_2^2}{2y_1 y_2}\right) - 1}\right) \\ &= \ln\left(\frac{y_1^2 + y_2^2}{2y_1 y_2} + \frac{-y_1^2 + y_2^2}{2y_1 y_2}\right) \\ &= \ln y_2 - \ln y_1. \end{aligned}$$

Answer 11. See the figures of Poincare disk.

Challenge Question:

1. Come up with your own tiling of regular polygon of the Poincare disk and Poincare hall-plane.

2. Consider the complex function $f : \mathbb{C} \sqcup \{\infty\} \to \mathbb{C} \sqcup \{\infty\}$ given by

$$f(z) = \frac{z - i}{z + i}.$$

a) Show that f maps the upper half-plane to the unit disk.
b) Show that f maps the geodesics of the Poincare half-plane to the geodesics of the Poincare disk.
c) Show that f, as a map from the Poincare half-plane to the Poincare disk, preserves distances.
d) Show that f preserves angles.

Note: This shows that the Poincare half-plane and the Poincare disk are in fact one and the same.