ZIGZAG PATHS and UPDOWN MOUNTAINS

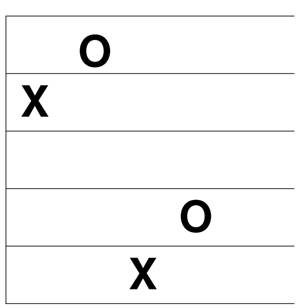
I. UPDOWN

On the next page are some game boards for a 2-player game called (you guessed it) **Updown!** You'll need to find someone to play the game with.

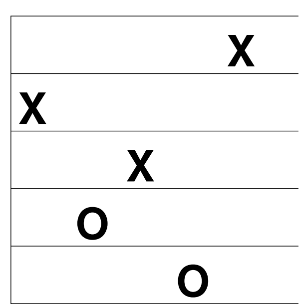
A game board has n rows, numbered 1 to n from bottom to top. The players $P1$ and $P2$ alternate turns, with $P1$ going first. A turn consists of putting an $\mathbf{X}$ ($P1$) or $\mathbf{O}$ ($P2$) in a row *that hasn't been used yet*. (Draw each mark a little further to the right, so you can see what has happened when the game is over.)

On the first turn $P1$ plays an $\mathbf{X}$ in any row. Then $P2$ plays an $\mathbf{O}$ anywhere else, in a row above or below the $\mathbf{X}$. If this $\mathbf{O}$ is above the $\mathbf{X}$, $P1$ puts a new $\mathbf{X}$ in any row below the $\mathbf{O}$; if the $\mathbf{O}$ is below the $\mathbf{X}$, $P1$ has to play above the $\mathbf{O}$. Play continues with each player moving in the direction (up or down) opposite to the previous player's move, and ends when one of the players cannot move. The other player is then declared the winner. (The game of course never lasts more than n moves, and may be shorter.)

Here are a couple of sample games with $n = 5$:



O wins



X wins

On the next page you can try $n = 4,\ 5,\ 6,$ and 7.

$n = 4$

$n = 5$

$n = 6$

$n = 7$

More sheets are attached at the end of this packet. You'll want to use some for the next part too.

Updown, tic-tac-toe, checkers, and chess are all examples of 2-person games *of perfect information*, meaning that each player knows all of the events of the game so far, as well as the conditions of winning or losing. (Card games where players hide their cards from one another are *not* examples of such games.)

Every 2-person game of perfect information and finite length, in which there are no ties or draws, *has a winning strategy* for either $P1$ or $P2$. (Updown fits this description. The other three just mentioned do not, since there are draws, but we could make them fit the description by declaring draws a win for $P2$.) That means that one of the players, if s/he plays perfectly (follows the winning strategy), will always win no matter what the other player does.

The explanation for this is simple: *can $P1$ play each move to ensure that (after this move) $P2$ doesn't have a winning strategy?* If yes, $P1$ has a winning strategy, since the game is finite. If no, then $P2$ has a winning strategy.

Problem 1. (a) Who has a winning strategy in Updown? Does your answer depend on n? What is the winning strategy?

(b) If you got (a), try answering the same questions if we change Updown so that the first player not to be able to move *wins* the game.

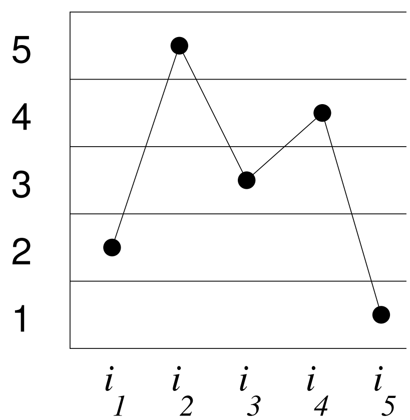
This question is reasonable for Updown or the draw-less Tic-Tac-Toe. For draw-less chess, though a winning strategy (for white or black) exists in theory, computing this is beyond the limits of current technology.

II. Mountains

An *updown sequence* of length n is a rearrangement $i_1, i_2, \ldots, i_n$ of the numbers $1, 2, \ldots, n$ such that

$$i_1 < i_2 > i_3 < i_4 > \cdots .$$

These are also called *alternating permutations*, but I prefer to call these sequences *mountains*. Here is a mountain of length 5:



visualizes $\underset{i_1}{2} < \underset{i_2}{5} > \underset{i_3}{3} < \underset{i_4}{4} > \underset{i_5}{1}$

Problem 2. How many updown mountains of length n are there (for small values of n)? Fill in as much of the table below as you can.

$n = 0$	$M_n = 1$
1	1
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	

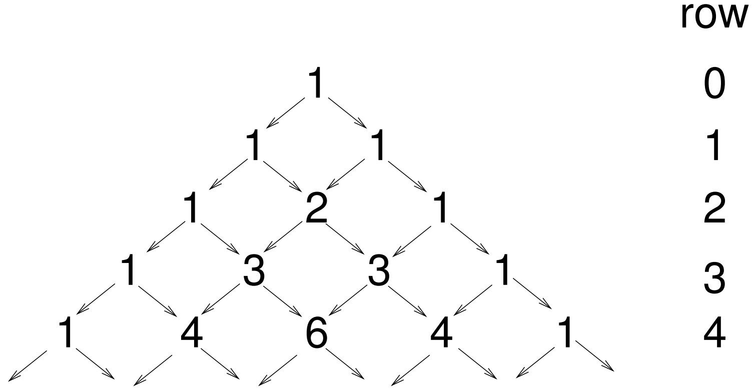
Note: the empty mountain is a mountain of length zero.

Use the sheets at the end of the packet to draw your pictures.

It's OK to only get up to n=5. You'll be able to fill in more of the table later.

III. ZIGZAG

Some of you have probably seen Pascal's traingle:

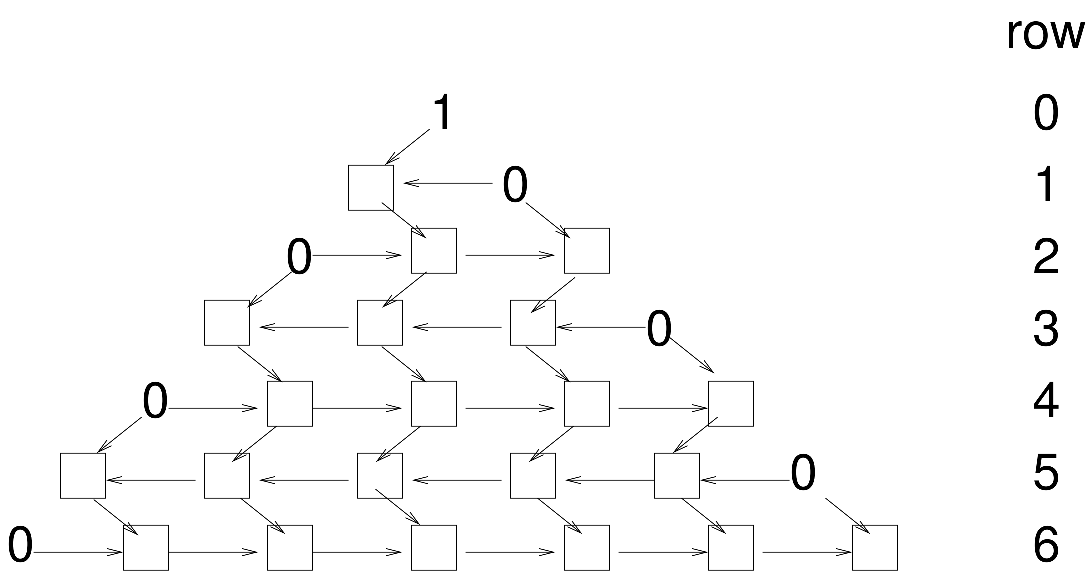


...and so on...

where the idea is *to figure out each entry by adding the (one or two) numbers from which arrows are pointing to that entry*. The k^{th} entry in the n^{th} row is denoted $\binom{n}{k}$ and tells you how many different ways you can choose k things from n things. I'm told that you will hear more about this next weekend.

What I want to do here is similar but different.

Problem 3. Fill in *Seidel's triangle*, using the same idea as in Pascal:

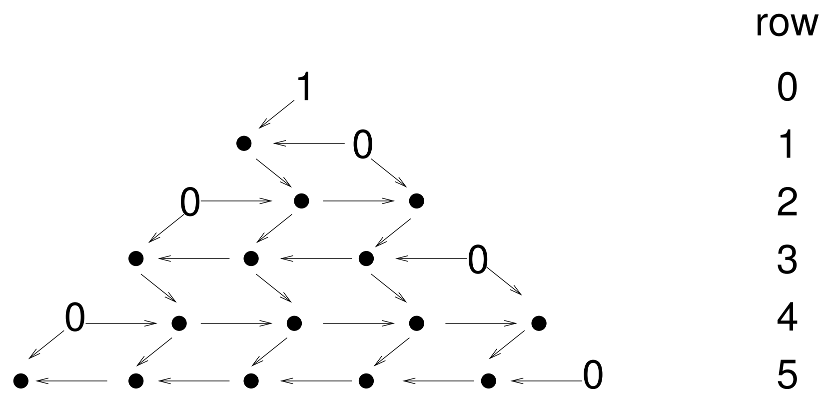


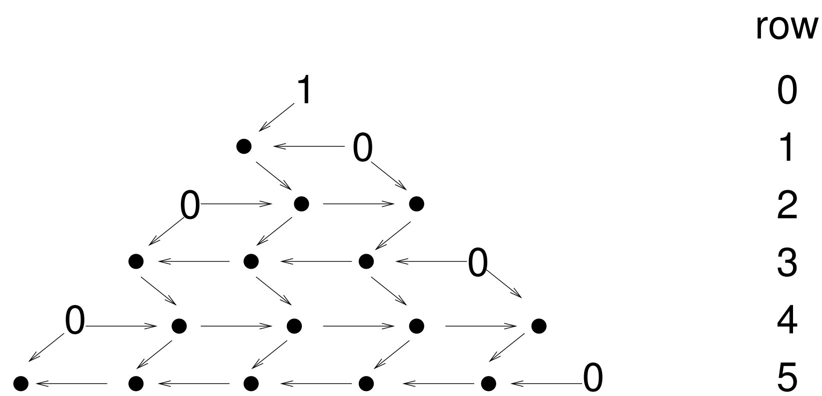
We will denote the k^{th} entry (in the direction of the arrows) in the n^{th} row by $\begin{bmatrix} n \\ k \end{bmatrix}$. What do you notice? Keep going if you like!

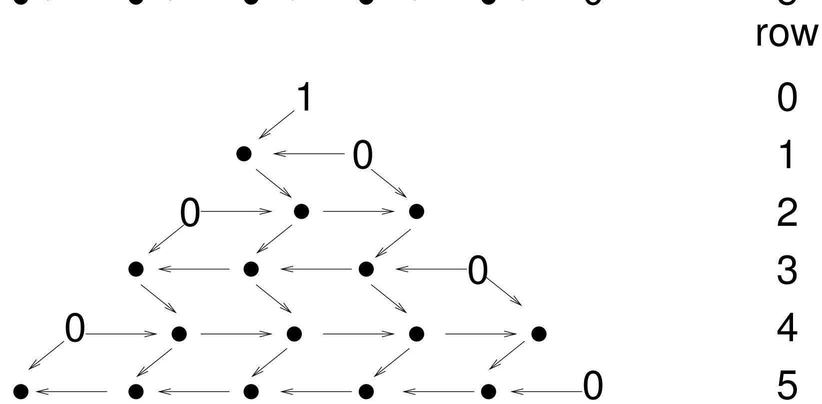
IV. PATHS

Here is another question about the Seidel zigzag triangle.

Problem 4. (a) Determine the number of paths, starting at the top and following the arrows, to your favorite entry in the triangle. What do you notice? Is there a reason for this? (Use the copies of the Seidel triangle below.)







(b) Write Z_n for the number of zigzag paths to the last dot in the n^{th} row. These are called *Euler*[1] *zigzag numbers*. Explain why $Z_n = M_n$, and use this to fill in more of the table in Problem 2.

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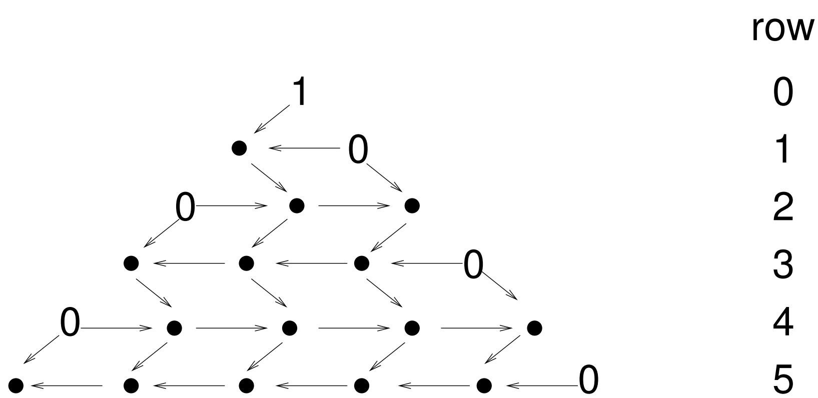
[1] or sometimes Bernoulli-Euler

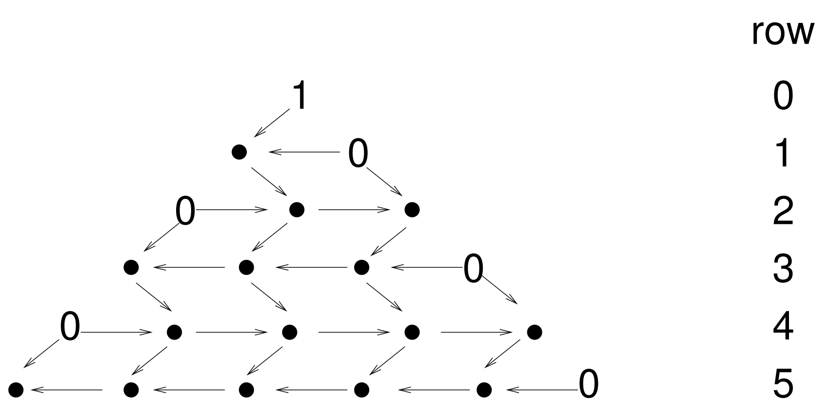
V. ZIGZAG MOUNTAINS?

If the number of zigzag paths Z_n equals the number of updown mountains M_n, then shouldn't there be some *natural* 1-to-1 correspondence between paths and mountains?

Problem 5. (a) Can you find a rule which, given a path to the last dot in the n^{th} row, produces an updown sequence (mountain) of length n?

[Hint: some more triangles are below. Start by numbering the dots according to their position in the row (1 to n), for *all* the rows. Then draw a zigzag path ending at the last dot in some row, and circle the dots where the path *enters* each row. Now follow the path in reverse and look at the numbers you have circled. This isn't yet an updown sequence but might suggest how to make one.]





(b) [optional] Figure out a way to go backwards from mountains to zigzag paths, so that you know you really have a 1-to-1 correspondence.

(c) [optional] Can you show that the number of paths to the k^{th} point in the n^{th} row is the number of updown mountains (of length n) starting with $i_1 \leq k$? or with $i_2 - i_1 \leq k$? (Both are in fact true.)

VI. ZIGZAG PIE?

The Euler zigzag numbers Z_n $(= M_n)$ come up in many parts of math, from trigonometry to number theory to singularity theory. They also have the feature of being easy to compute, using the Seidel traingle.

Here's one thing you can do with them. Make two lists using what you've filled out of the table in Problem 2.

n =	2	4	6	8	10	12
$(2n+2)Z_n$ =						
Z_{n+1}=						

Now, in the last row, write the quotients $(2n+2)Z_n/Z_{n+1}$ of the numbers above. (You will need a calculator for this part.) What do these fractions appear to get close to?

$n = 4$

$n = 5$

$n = 6$

$n = 7$

$n = 4$

$n = 5$

$n = 6$

$n = 7$

$n = 4$

$n = 5$

$n = 6$

$n = 7$

$n = 4$

$n = 5$

$n = 6$

$n = 7$