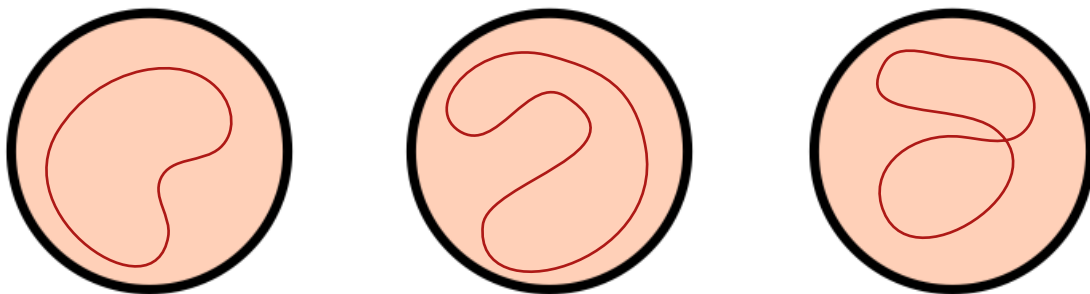# Loops and the Winding numbers

1. Loops

 Today, let's study the various movements of loops in some geometric figures! In this talk, we treat loops in some specific figures. **Firstly, let's consider loops in a disk**. Here, a "loop" in a disk is a deformable, very soft, closed curve like the following figures:

Fig.1

Note that **a loop can cross with itself** (the right figure). To draw richer information, we define the **direction** on each loop. There are only two type directions: the clockwise direction and the anti-clockwise one.
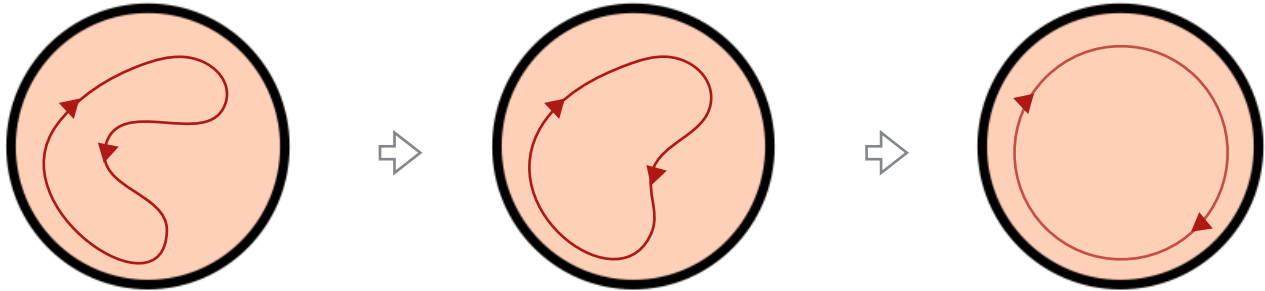
Fig.2
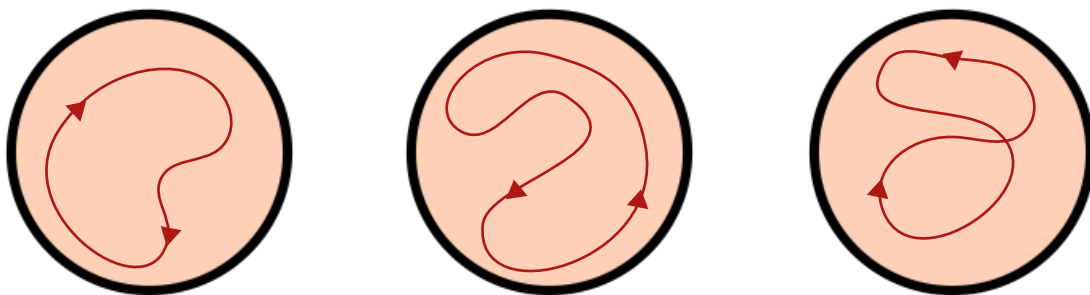
Q.1
In the Fig.2, which direction does each loop have?

Now, let's modify this soft loops under some rules!

Example.3

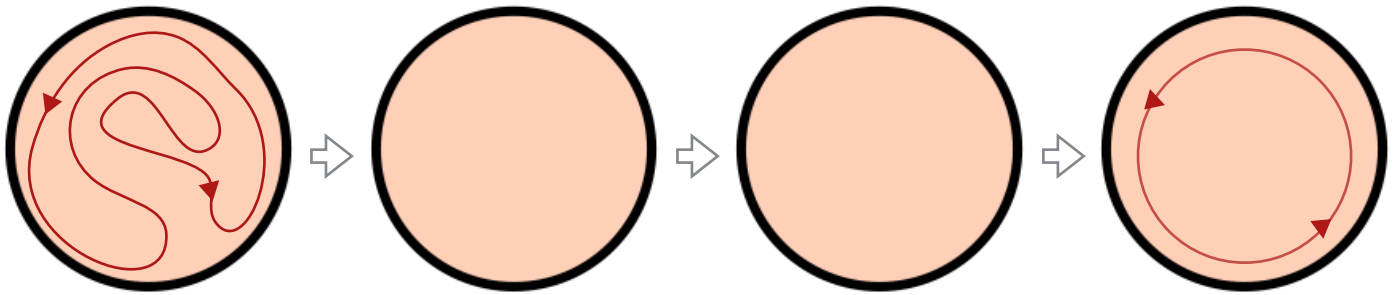This modification is quite free, but there are two restrictions:

Fig.4
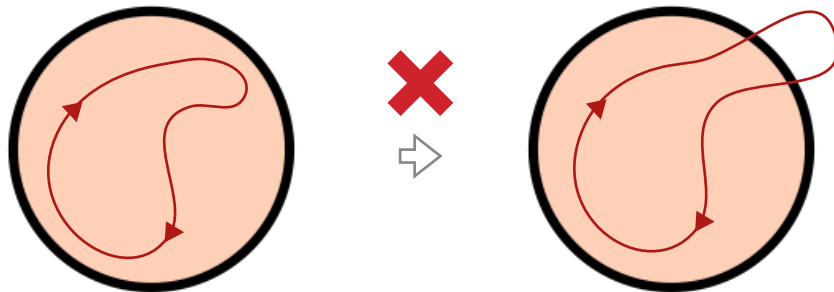
Fig.5
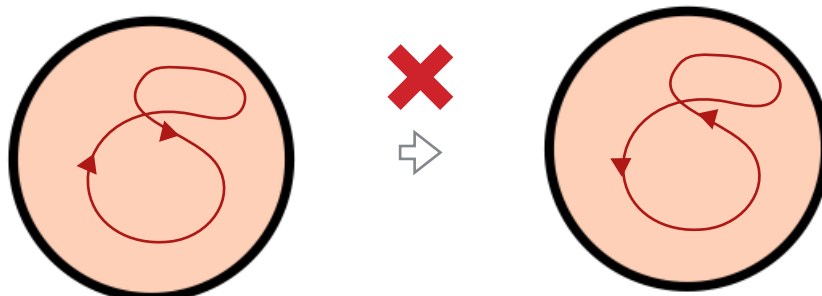
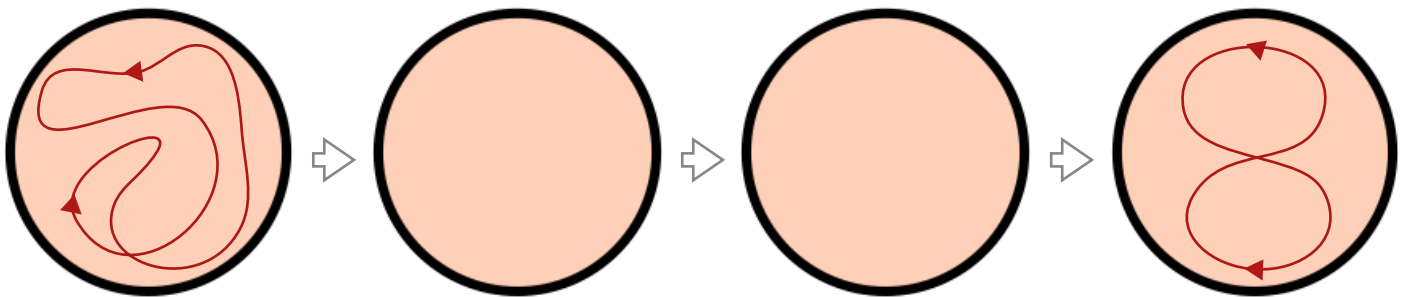Under these three rules, let's modify loops in various figures!

Q.2

Try drawing the modifications of the following loops in the middles
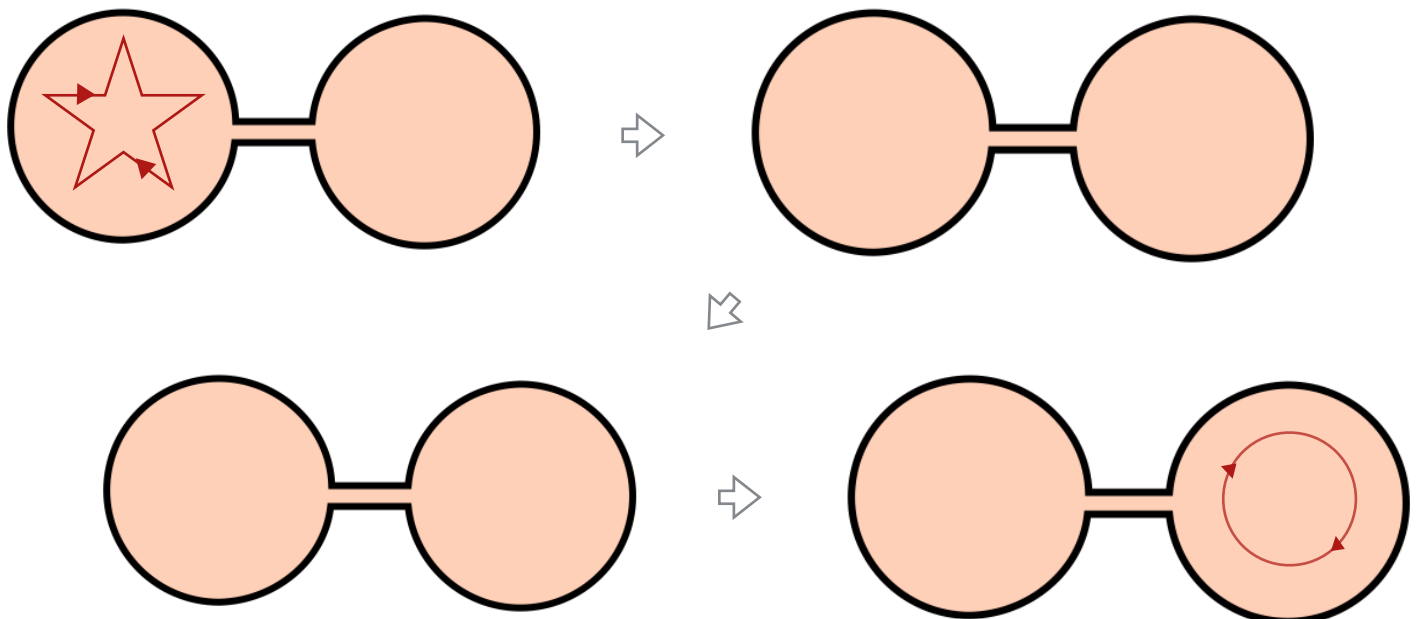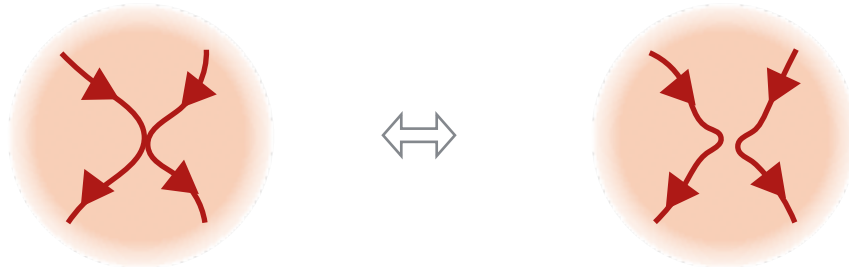
1.

2.

3.

Next, we treat crossing points on a loop. By splitting a loop at its crossing point, we can decompose it to some loops. As the opposite operation, we can also stick some loops to just one loop.
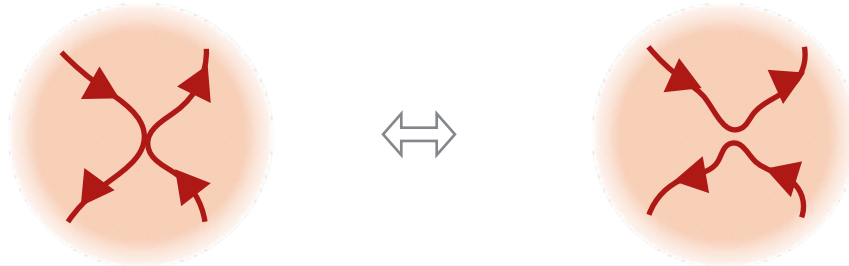
Rule 4

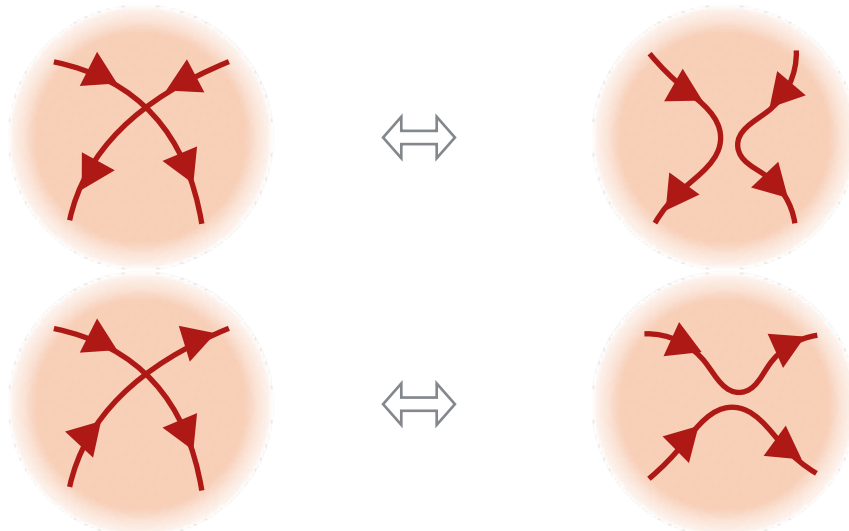Tangent points can be split and stuck as follows:

Case1.(same direction loops)
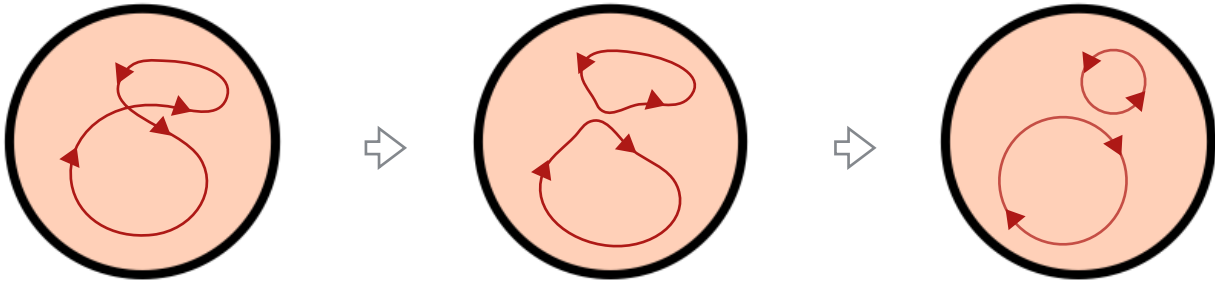

Case2.(opposite direction loops)

Q.3

By using Rule4, we can also obtain the following two modification rules about the crossing points. How should we use Rule4?
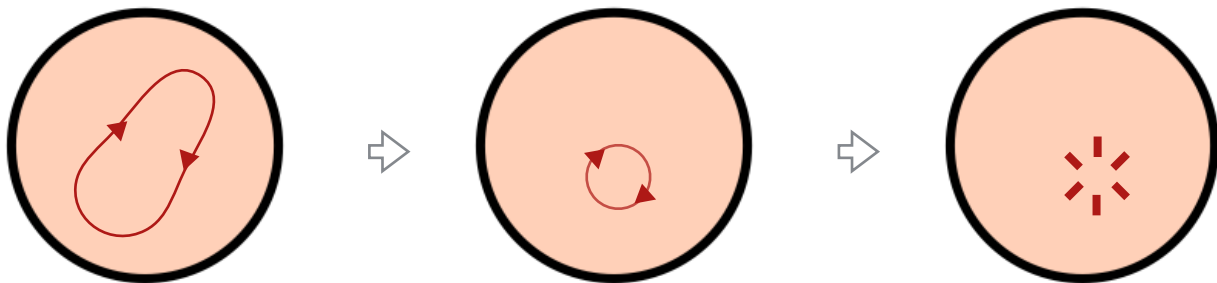
Example.6

By using these rules, any complicated loops in the disk can be modified to very simple circles.
 Now, we have reached the final rule!

Rule 5
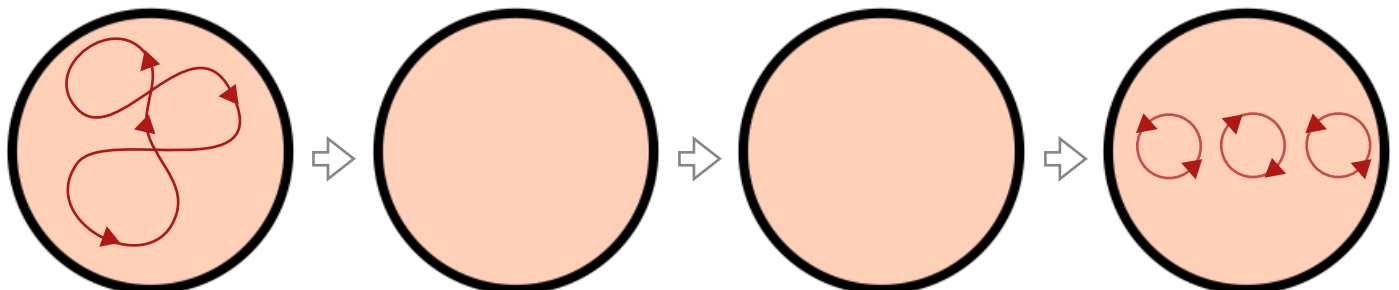   If loop can be shrunk to any small circle, we can erase the loop.
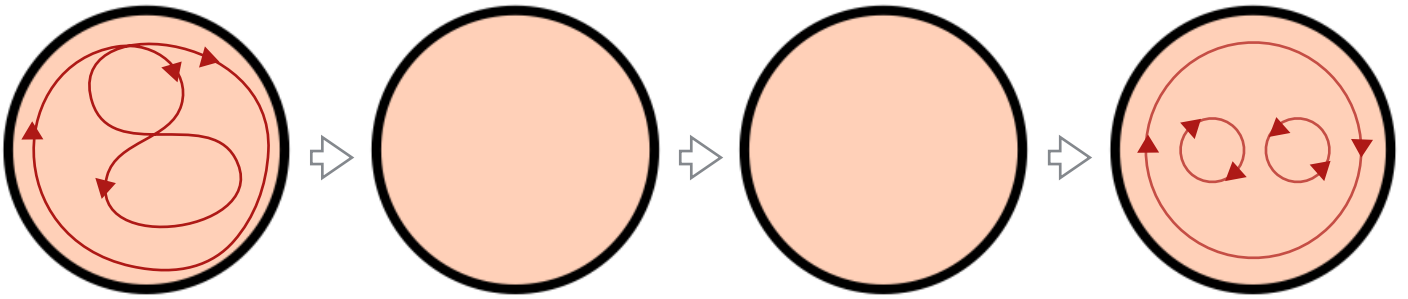
Example.7

Q.4
     Try drawing the modifications of the following loops in the middles. For 4. ~ 7. , try erasing the given loops.
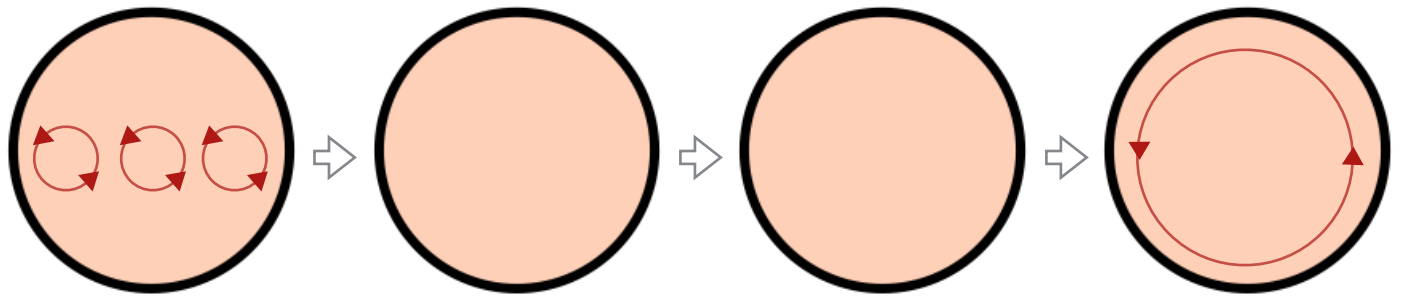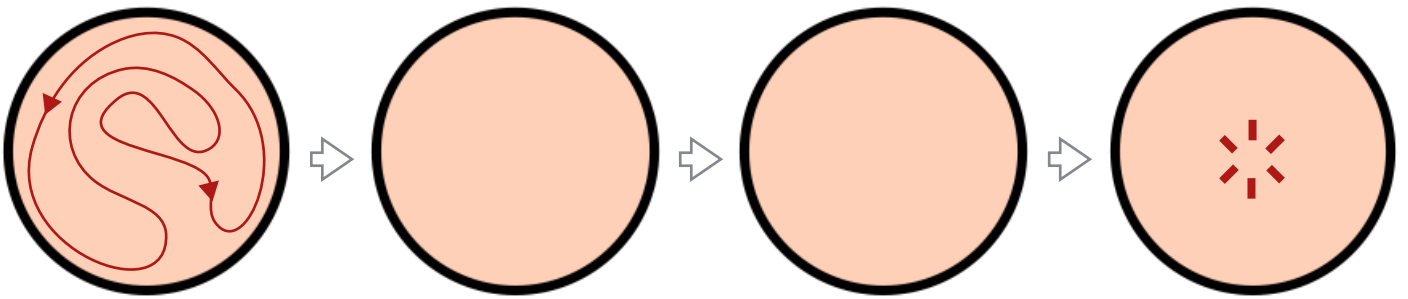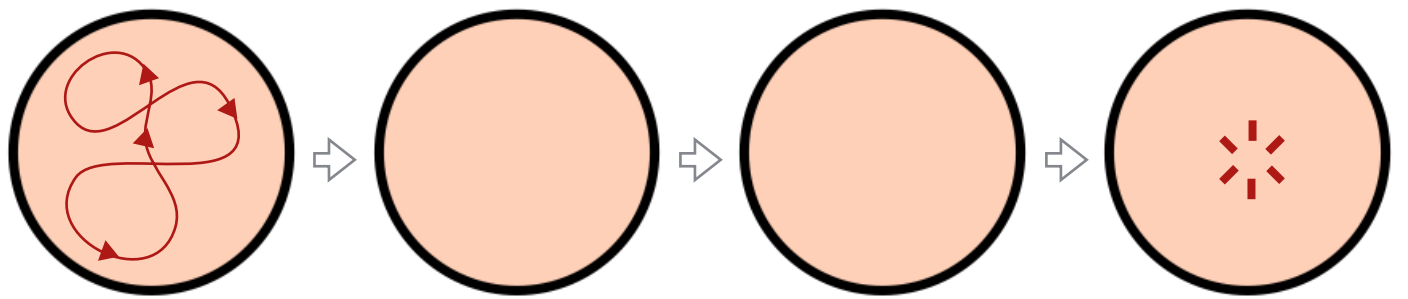1.

2.

3.

4.

5.

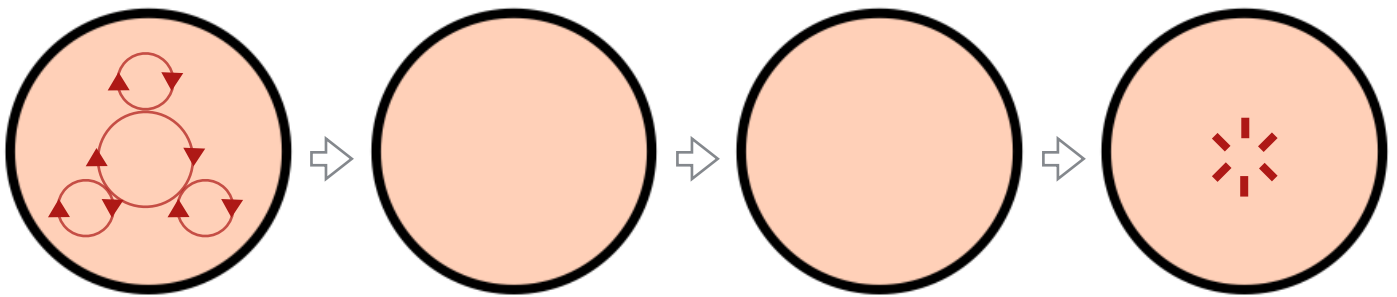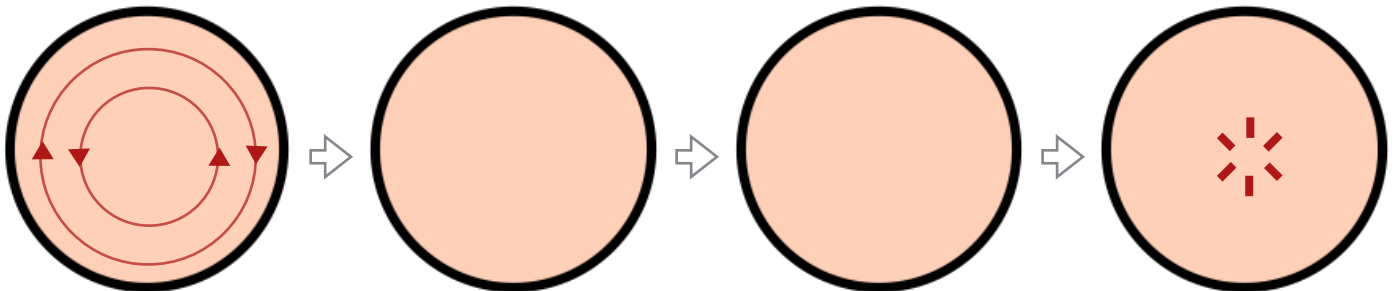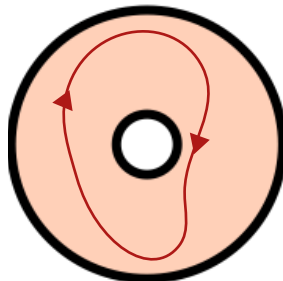

6

<u>6.</u>

<u>7.</u>

Fact: Any loops in a disk can be erased

2.Winding number

In the section 1, we have seen a lot of loops in a disk. However, all of them can be shrunk to a point and erased⋯this is a very good geometric property of the disk. By considering more complicated figures instead of a disk, **we can find a loop which cannot be erased** such as the following one:

Fig.8

Because of the central hole, we cannot shrunk this loop to a point. There are a lot of examples of figures which has such a "unerasable" loops.

Q.5

In the following figures, which ones do have unerasable loops? Try drawing such a loop.

(a)

(b)

(c)

(d)

(e)

(f)
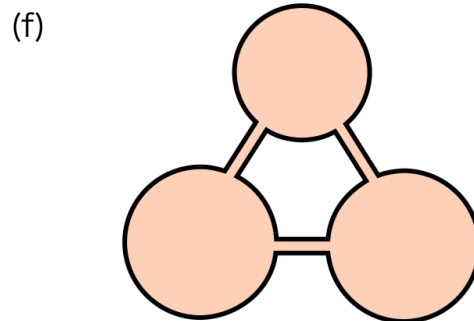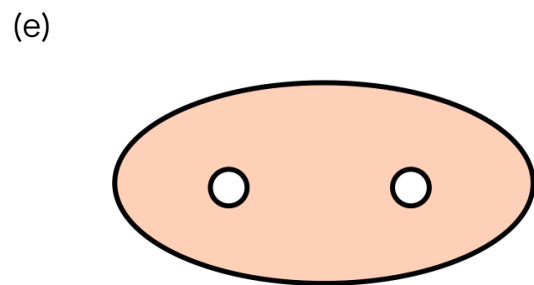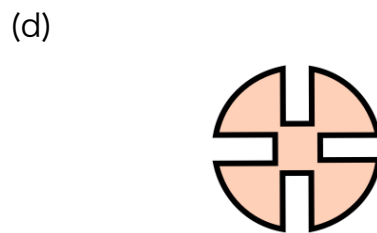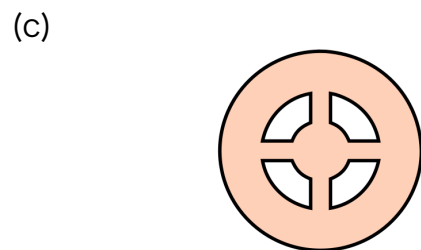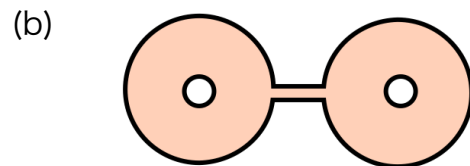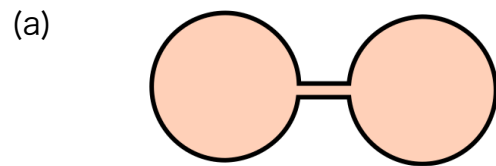
(g) (The figure which is defined by getting rid of just one point from a disk)
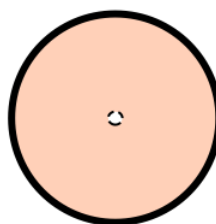
Let's go back to the disk with the central hole. Though we cannot erase some loops to points in this figure, we can modify them to simple circles:

Example.9
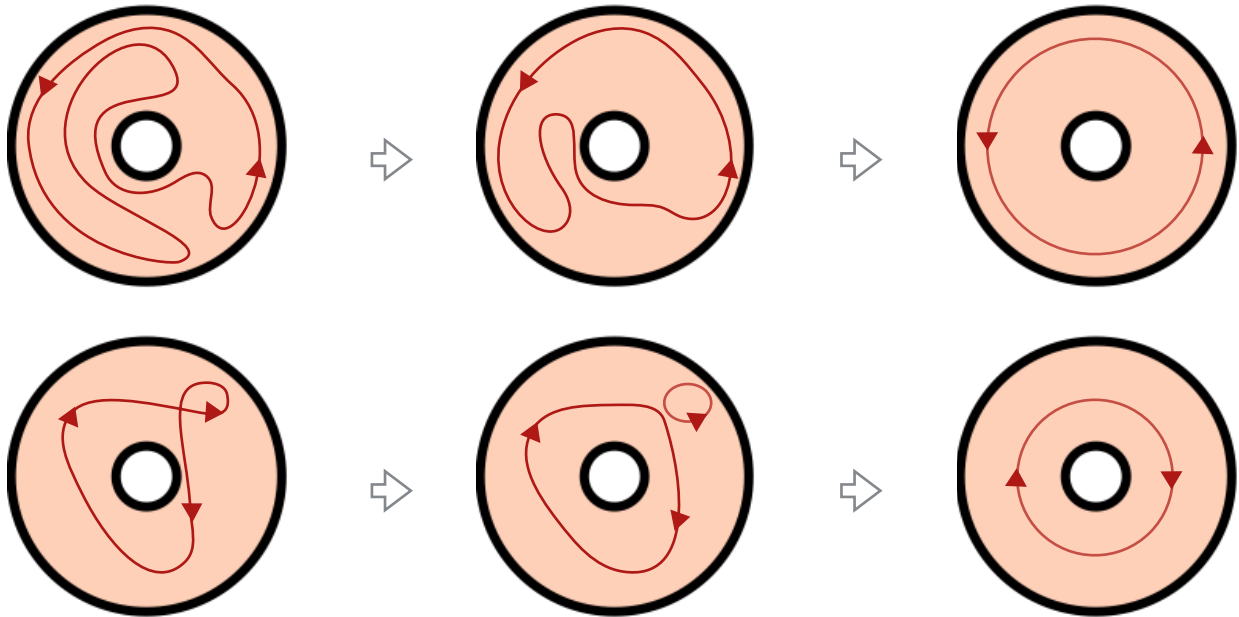
Q.6
Try drawing the modifications of the following loops in the middles

1.

2.

3.
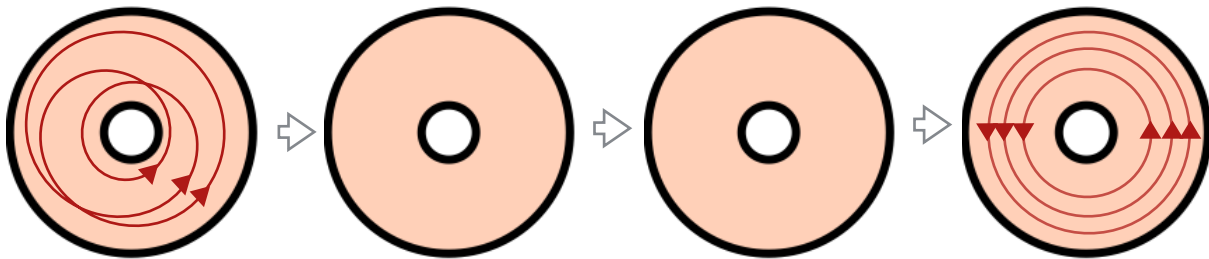
4.
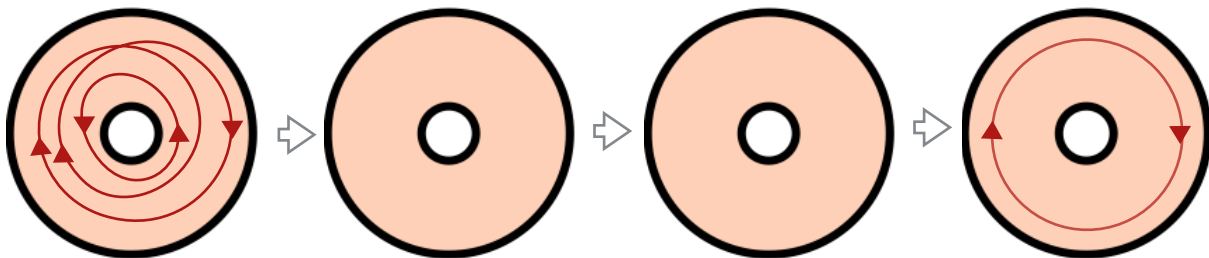
Fact: Any loops in the disk with a hole can be modified to n-number of circles with the same direction:

Case1.
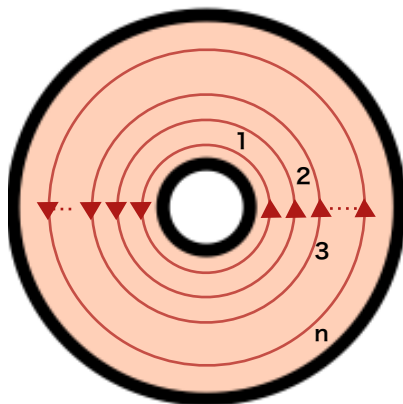
Case2.
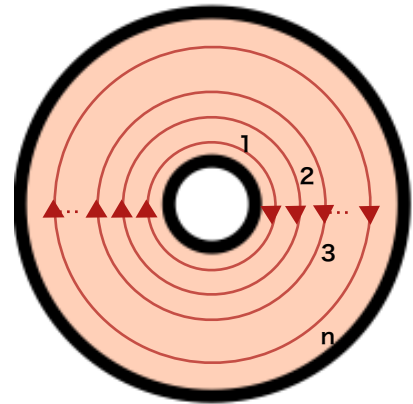
If a combination of loops can be modified to n-number of anti-clockwise circles, we say that **the winding number of the loops is n.** If it can be modified to n-number of clockwise circles, we say that **the winding number of the loops is -n.**
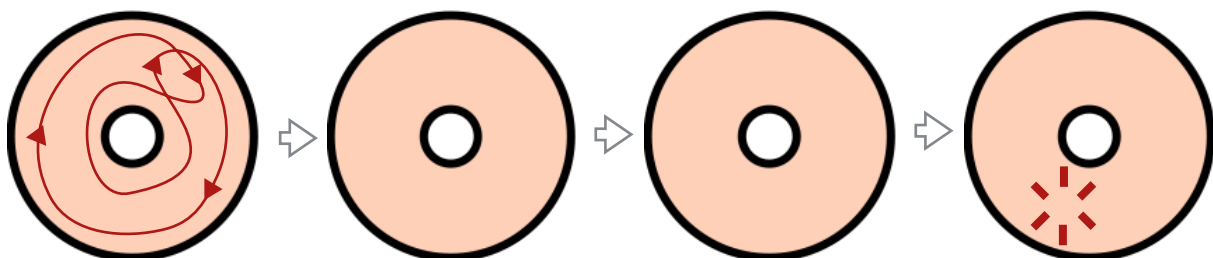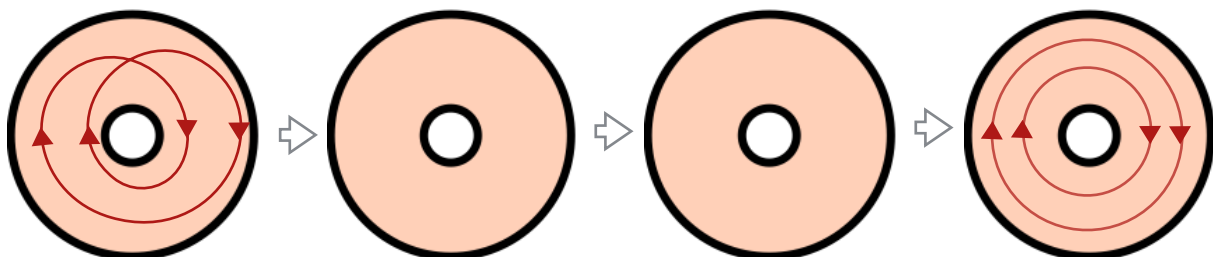
Example10.

In the Q.6, we have seen the modifications of given loops. From them, we can count the winding number of each of them:

1.   2.   3.   4.

n = 0   n = -2   n = 3   n = -1
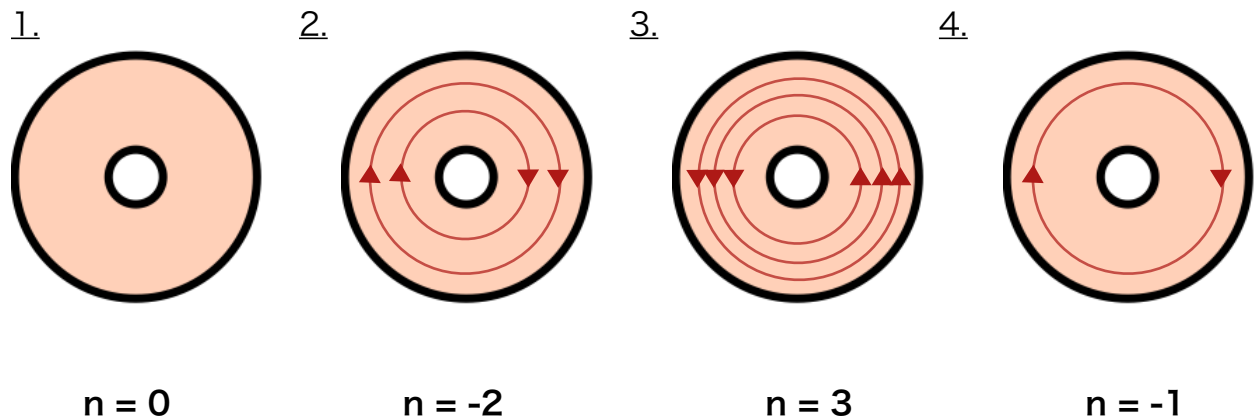
Q.7

Try computing the winding number of the following loops.

(1)   (2)   (3)

(4)   (5)

11

Why do we want to know the winding number of loops? One reason is that **two (combinations of) loops have the same winding number if and only if each of them can be modified to the other.** Hence, by using this number, we can classify each loop with respect to the modification

Fig11.

n=1 loops

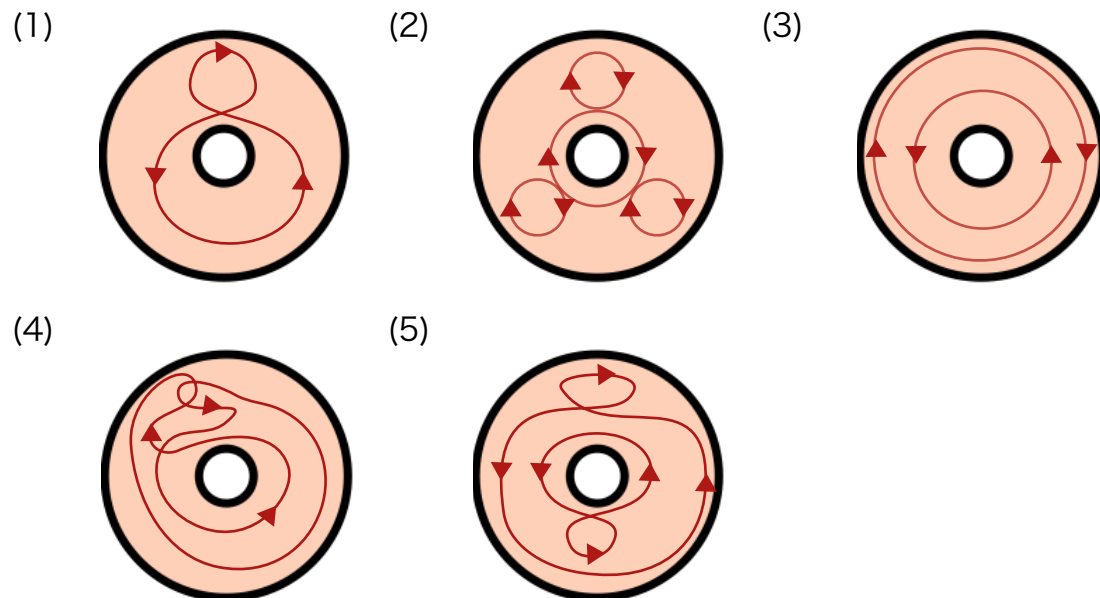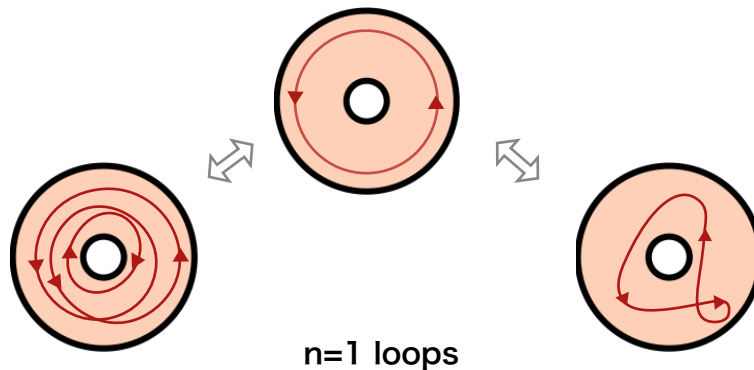To use this very useful application, we want to know **how to count the winding number quickly**. It is given by the following procedure. Firstly, draw a vertical line which connects the boundary of the disk and that of the hole:

Fig12.
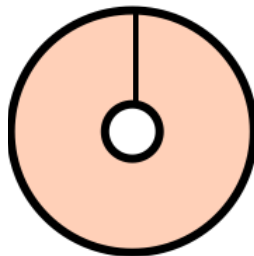
Next, for the given loop, count the number of crossing points with this line. Here, we have to be careful about the direction of the loop. We count each crossing point with the sign +1 or -1:

Fig13.

+1   -1

Then, by summing up the crossing points, we can obtain the winding number!
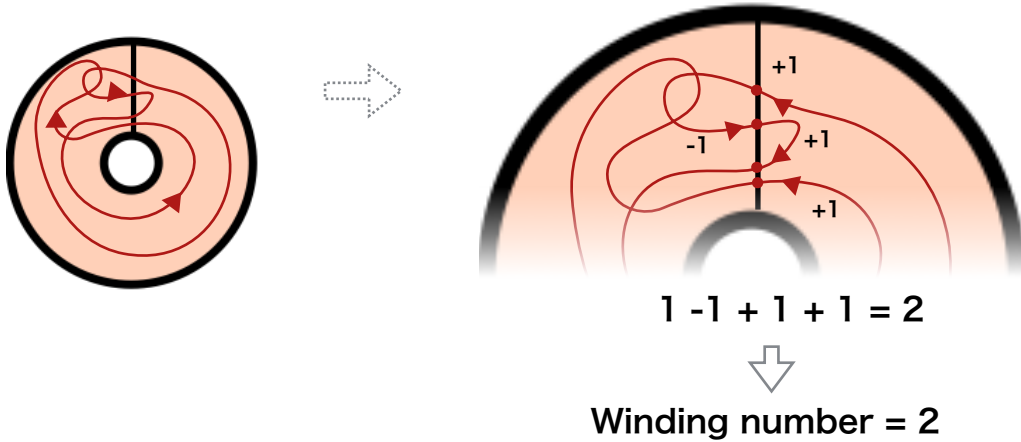

Formula

    (Winding number)   =   (Number of crossing points with the sign)

Example14.

1 -1 + 1 + 1 = 2

Winding number = 2

Q.8

      Try counting the winding number of loops in Q.7 by using the formula.
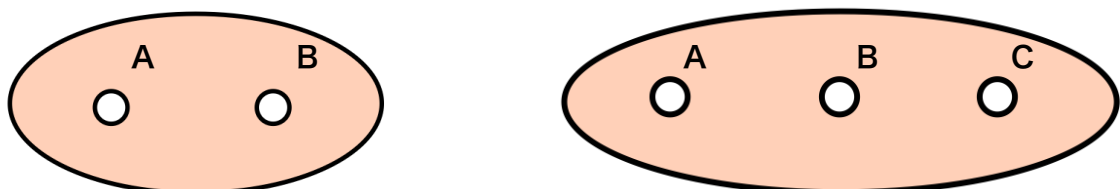
Q.9

      Drawing your own loops on the work sheet and counting their winding numbers by the formula. Now, you can classify the loops very quickly!

Q.10 (Advanced topics)

      We can consider the winding number even if the figure has more than one holes. Let's consider the case of two and three holes. Let's call these hall A, B, (and C).
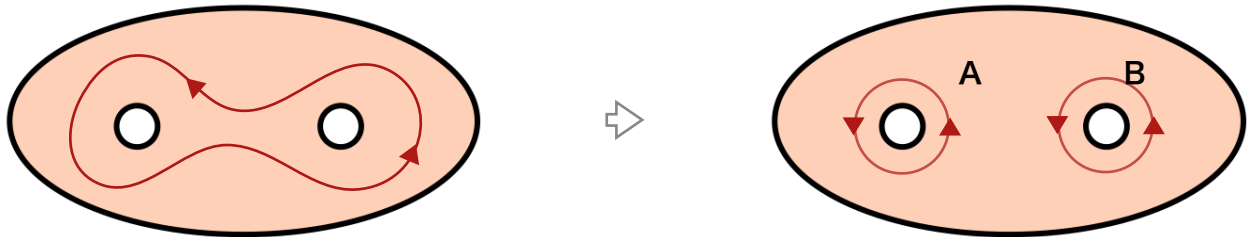
Fig.15

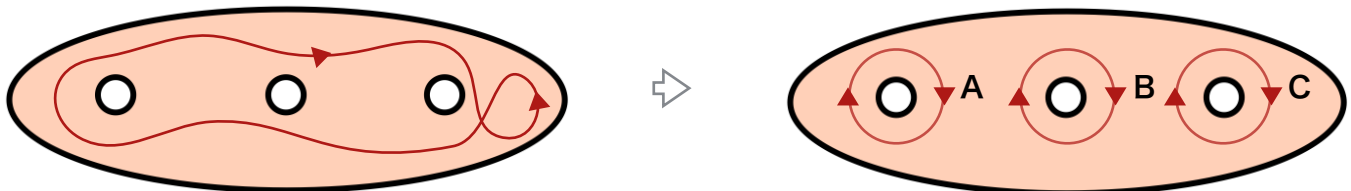In this case, instead of n-number of circles, **we can modify any loop to a combination of n-numbers of loops around A, B, (and C) with the sign**. Let's check this result for some cases in the following question:

(a) Check the modifications of the following loops.

<u>1.</u>
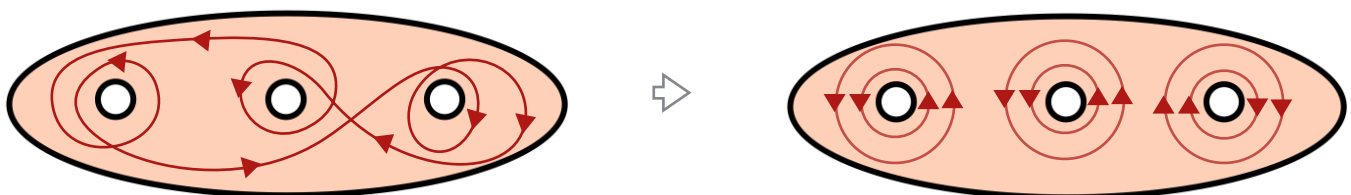
<u>2.</u>

<u>3.</u>

<u>4.</u>

The winding number at each of A, B, (and C) is defined by the number of these circles with the sign. In (a), each loop has the winding numbers as the following list:

1.   1 at A, 1 at B

2.   -1 at A, -1 at B, -1 at C

3.   -2 at A, -1 at B

4.   2 at A, 2 at B, -2 at C

(b) Like the case of the disk with just one hole, try finding the general formula to compute the winding numbers for the disk with two or three holes. (Hint: drawing some lines…)
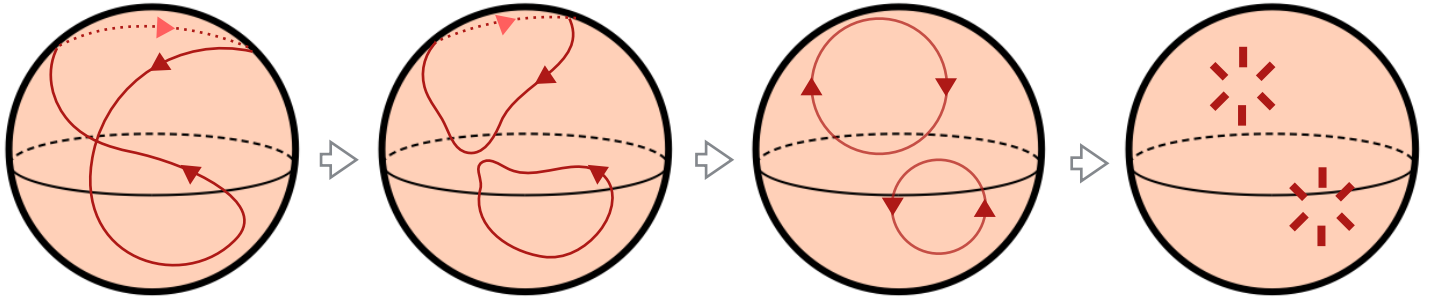
3. Loops in Solid Figures.

We have seen the modification of loops in the flat 2-dimensional figures. What will be happened if we jump up to 3-dimensional figures? Even for loops in the surface of a solid figure, we can apply the same rule to observe the modification. As the first step, let's consider the loops in the surface of the sphere.

Fig.16
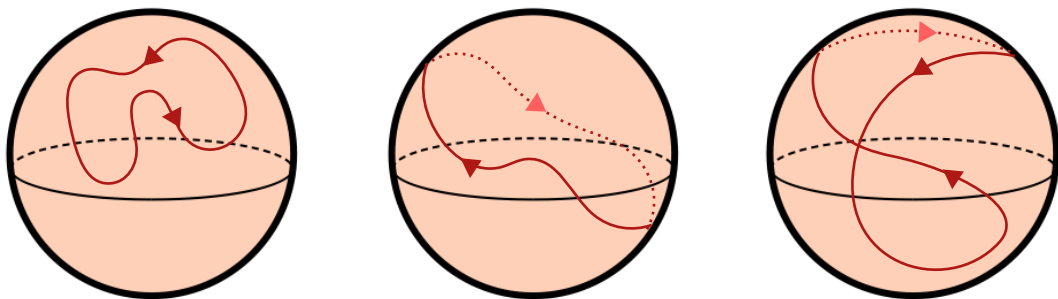
Fact: Any loops in a sphere can be erased

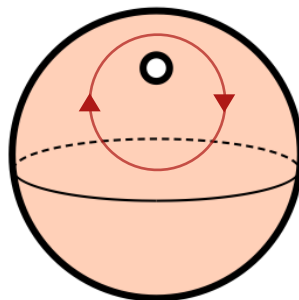Let's see the vanishing of the loop in the right picture of Fig.16:

Q.11

Check that we can erase the other two loops in Fig.16 via modifications.

What a kind of solid figure can have unerasable loops? How about giving a hole to the sphere?
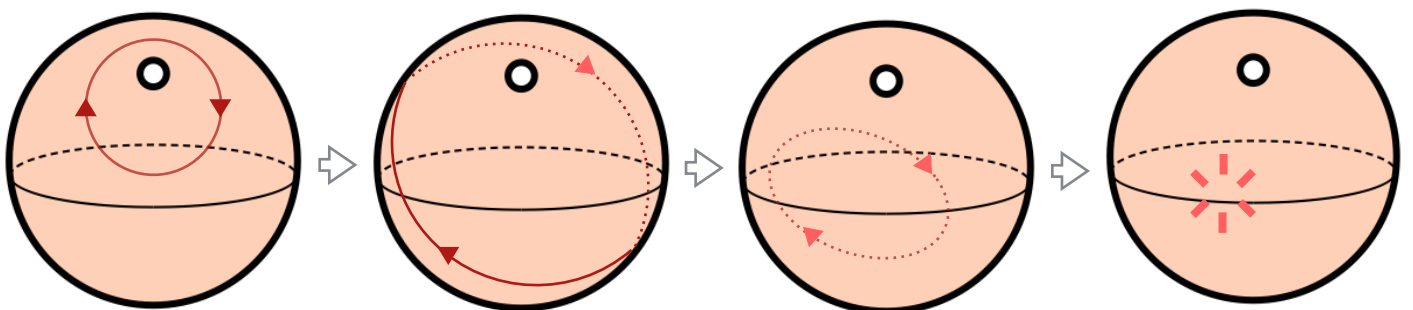
Fig.17

Unerasable loop?

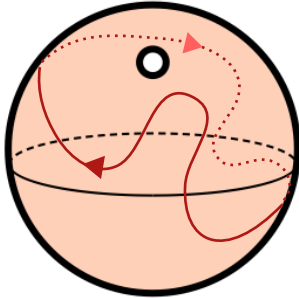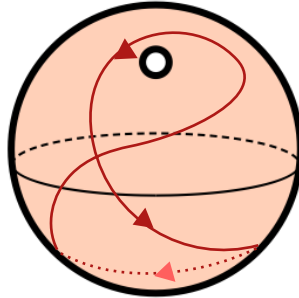Unlike in the case of the flat figures, we can erase this loop!

Fig.18

Q.12

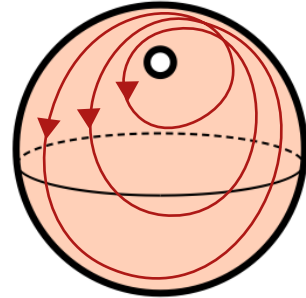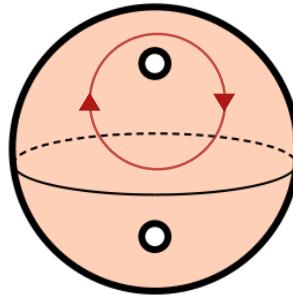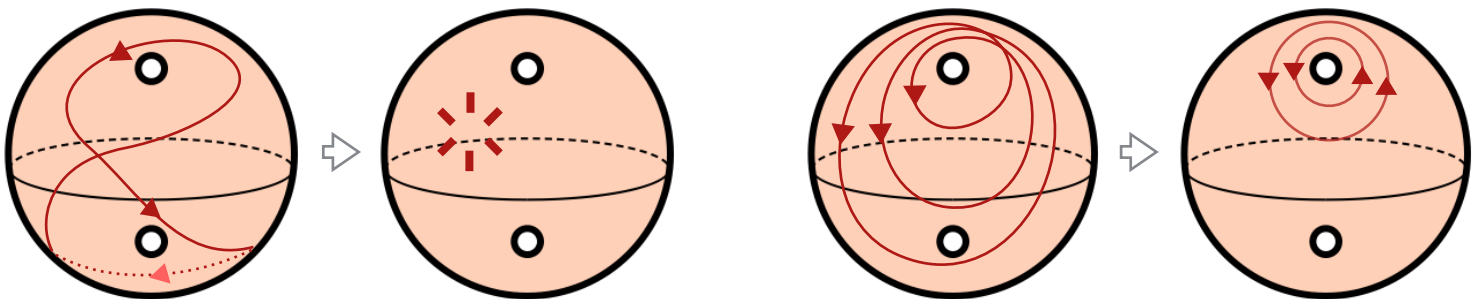**Check that we can erase the following loops.**

1. 2. 3.

One way to get an unerasable loop is giving two holes to the sphere:
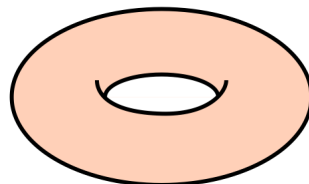
Fig19.

Then we can obtain the classification of the loops in this figure, similarly to that by the winding number.

Fig.20

Another way is considering the surface of the figure like a "donut". In mathematics, we call this kind of figures **(2-dimensional) torus.**
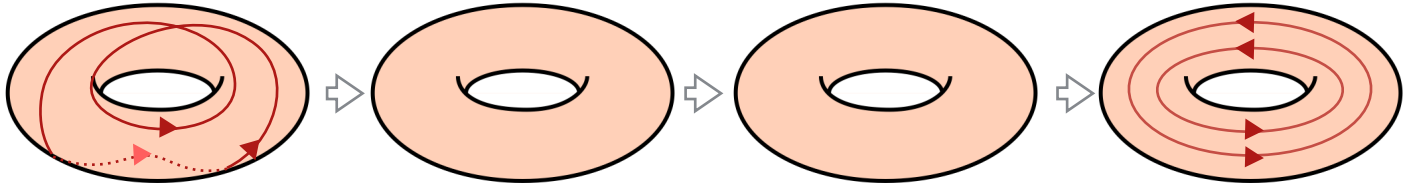
Fig.21

Torus

Through the following question, let's observe the classification of loops in the torus.
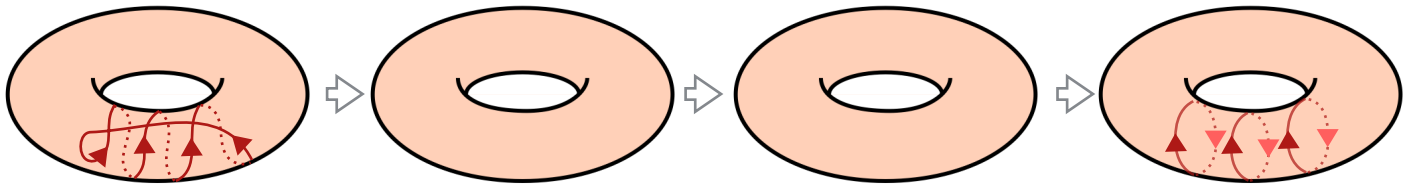
Q.13

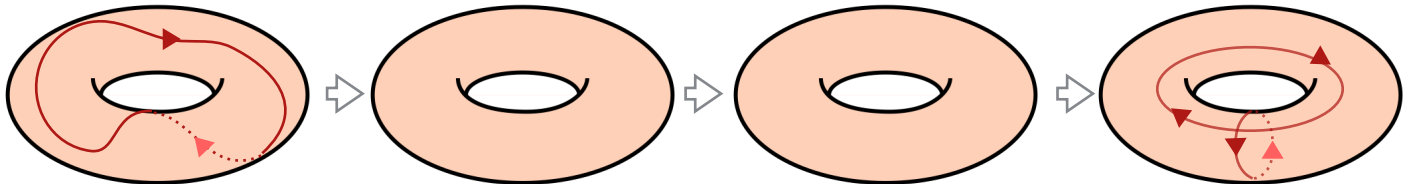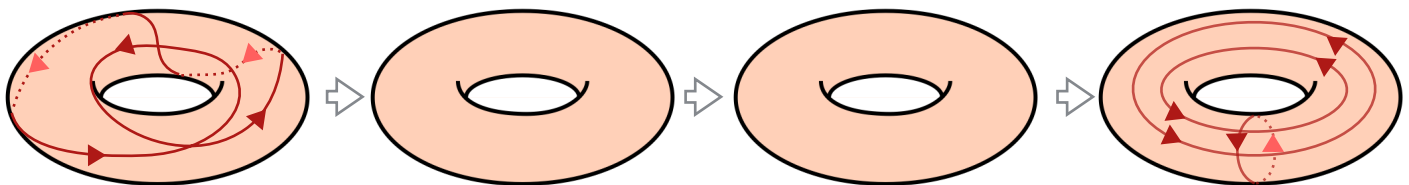**Try drawing the modifications of the following loops in the middles**
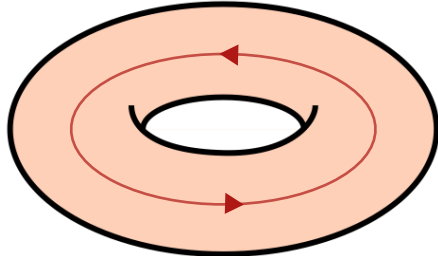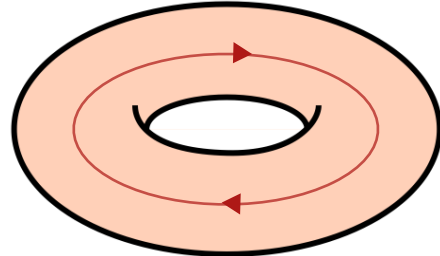
1.

2.

3.
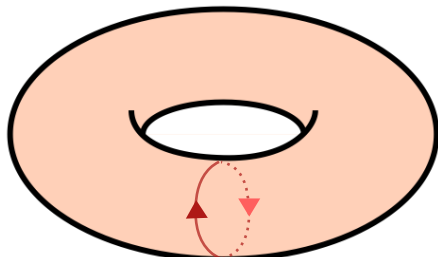

4.

These examples are evidence of the following result.

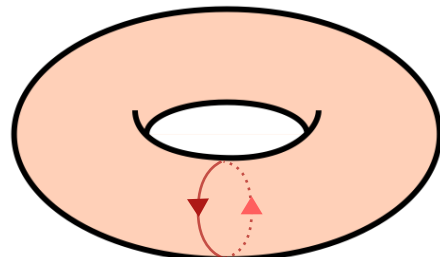**Fact: Any loops in the torus can be modified to a combination of the following four loops.**

+1 α

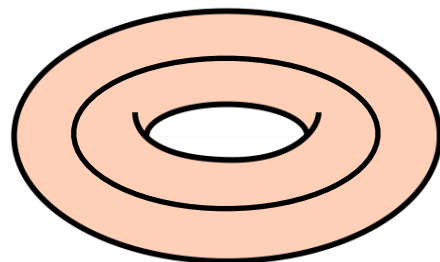-1 α

+1 β

-1 β

Like the winding number, by using α and β, we can classifying the loops in the torus. In the **Q.13**, the loops are classified to

    <u>1.</u>   ($+2\alpha$, 0)
    <u>2.</u>   (0, $+3\beta$)
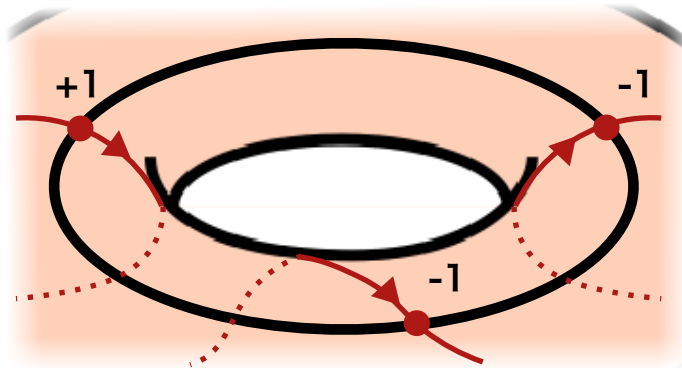   <u>3.</u>   (-1α, -1β)
    <u>4.</u>   ($+2\alpha$, -1β)

Moreover, even in this case, **we can compute the number of α and β very quickly.** Draw a circle for each of α and β.

19
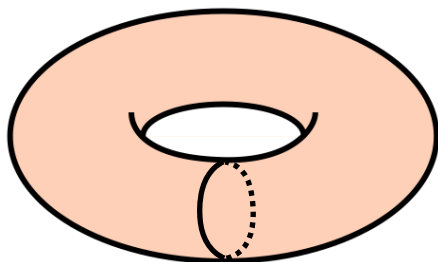
<u>The numbers of α and β are given by counting the numbers of crossing points (with the sign) with these circles.</u>
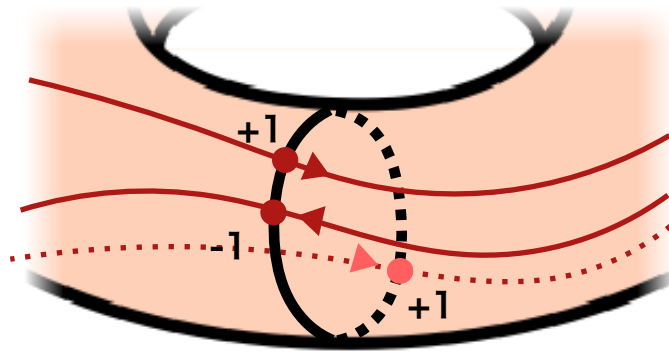
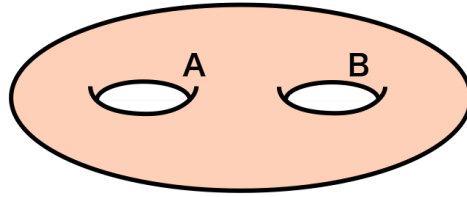+1-1+1 = +1 ⇨ +1 α

+1-1-1 = -1 ⇨ -1 β

<u>Q.14</u>

Check this computation for the loops in Q.13

<u>Q.15</u>

Drawing your own loops in the torus and classify it with respect to α and β.

Q.16 (Advanced topics)

      Instead of the torus, let's consider the following figure (it is called **genus 2 torus**):
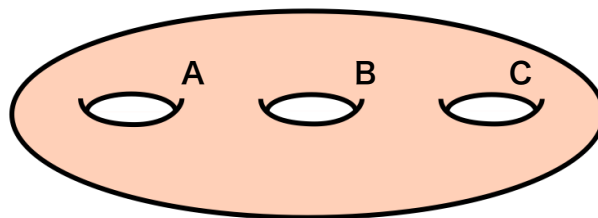

genus 2 torus

Any loops in this figure can be classified by using fundamental 8 loops

$$\pm 1\,\alpha \text{ at A,} \quad \pm 1\,\beta \text{ at A,} \quad \pm 1\,\alpha \text{ at B,} \quad \pm 1\,\beta \text{ at B.}$$

(a) Try guessing which loops we should take as these 8 types.

(b) Try finding the method to compute the number of each α, β at A, B (Hint: drawing some circles···).

(c) Try thinking the same classification for genus 3 torus:

genus 3 torus