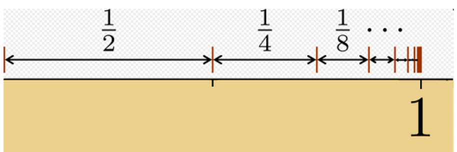
Today we are going to have fun with integers, primes, fractions and infinite series. Turn the page if interested!

1. What do you get if you add a small number, say $\frac{1}{2}$, infinitely many times? In notations, what is the infinite sum $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$?

2. What do you guess about the infinite sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$?

   [Hint: You could think about successively bisecting a one-meter stick. The following picture might help.]



3. This activity is OPTIONAL, especially suggested for higher grades. What do you guess about the following infinite sum?

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+\cdots$$

[Answer: The series grows without bound. One way to justify this is by grouping this bunch of infinitely many small numbers into groups each group greater than $\frac{1}{2}$.]

Our general goal today is to convince ourselves that,

$$\boxed{\sum_{p \ \text{prime}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = \infty}$$

To do this we need develope some experience with primes and counting techniques.

4. Find the primes less than or equal to 100. My suggestion is to strike out those numbers in the following list which are NOT prime.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	83	94	95	96	97	98	99	100

5. Primes are multiplicative building blocks of integers, I mean, every integer (except for 1) can be written uniquely as product of primes; for example $12 = 2^2 \times 3$. Do the same for $18, 21, 77, 100$.

6.

- How many integers $\leqslant 100$ are mulpliers of 2?

- How many integers $\leqslant 100$ are mulpliers of 3?

- How many integers $\leqslant 100$ are mulpliers of 7? An upper bound suffices.

- How many integers $\leqslant N$ are mulpliers of k, where k is an arbitrary integer? An upper bound suffices.

- How many integers $\leqslant N$ are divided by 2 or 3? An upper bound suffices.

- How many integers $\leqslant N$ are divided by p or q, where p and q are two distinct arbitrary primes? An upper bound suffices.

- How many integers $\leqslant N$ are divided by p or q or r, where p, q, r are three distinct arbitrary primes? An upper bound suffices.

7. - Suppose you have two shoes and three trousers which all match nicely together. In how many different ways can you wear them?

   - Suppose you have two shoes, three trousers and three shirts which all match nicely together. In how many different ways can you wear them?

   - Do you agree with the following general law?

     > If there are m ways of doing something and n ways of doing another thing, then there are mn (product of m with n) ways of performing both.

     This is usually called *the multiplication principle*.

For natural number k, let us denote the k–th prime by p_k, for example

$$p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7, \cdots .$$

Now we set out to prove

$$\sum_{p\ \text{prime}} \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \cdots = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \infty.$$

We argue by contradiction, that is, we assume our sum to be finite and look for something absurd! Since now our sum is supposed to be finite its tail constantly decreases to zero, so for some big integer k,

$$(k\text{–th tail}) = \frac{1}{p_{k+1}} + \frac{1}{p_{k+2}} + \frac{1}{p_{k+3}} + \cdots < \frac{1}{2}.$$

Fix this number k somewhere in your mind. Based on our intuition, let us call primes $2, 3, \cdots, p_k$ *small*, and call the rest $p_{k+1}, p_{k+2}, p_{k+3}, \cdots$ *big*. Turn the page if you are ready!

Fix in your mind a large number N, say $N = 100$. Consider all integers less than or equal to N. These are of two kind: either they are merely composed of small primes, or at least one big prime appears in their composition. Suppose we have S numbers (S for Small) of the first kind and B numbers (B for Big) of the second kind. Thus

$$N = S + B.$$

8. Estimate B, that is find a reasonable upper bound for the number of integers less that or equal to N which are divided by p_{k+1} or p_{k+2} or p_{k+3} or ... .

   [Hint1. Use the formula you found in the sixth activity.]

   [Hint2. If this seems too hard for you, just try to convince yourself that the following estimation works,

$$B \leqslant \frac{N}{p_{k+1}} + \frac{N}{p_{k+2}} + \frac{N}{p_{k+3}} + \cdots \leqslant \frac{N}{2}$$

   .]

9. Estimation of S.

- An integer $A \leqslant N$ of the first kind, that is an integer whose all prime farctors are small is (uniquely) of the form,

$$A = 2^* \times 3^* \times \cdots \times p_k^*,$$

where $*$ denotes an arbitrary exponent. Gathering all prime factors with even exponents, A can be written as

$$A = R^2 \times 2^{0 \text{ or } 1} \times 3^{0 \text{ or } 1} \times \cdots \times p_k^{0 \text{ or } 1},$$

where now each exponent is either 0 or 1.
Constructing such an integer A is a $k+1$-step procedure, one step for choosing R and k steps for choosing exponents (0 or 1).
  - How many choices do you have for $2^{0 \text{ or } 1} \times 3^{0 \text{ or } 1} \times \cdots \times p_k^{0 \text{ or } 1}$?
    [Hint. Use Multiplication Principle, you have learned in sixth activity.]
  - Note that $R \leqslant \sqrt{A} \leqslant \sqrt{N}$. How many choices do you have for R?
    [Hint. Again Multiplication Principle.]

- Estimate S from above.
  [Hint. Again Multiplication Principle.]

From our estimations,

$$N = B + S < \frac{N}{2} + \sqrt{N}\, 2^k.$$

Thus $N < \sqrt{N}\, 2^{k+1}$. Therefore $N < 4^{k+1}$. This is our desired contradiction.

Reference.

Martin Aigner and Günter Ziegler, *Proofs from THE BOOK*, Fourth Edition, Springer, 2010, pp 5-6.