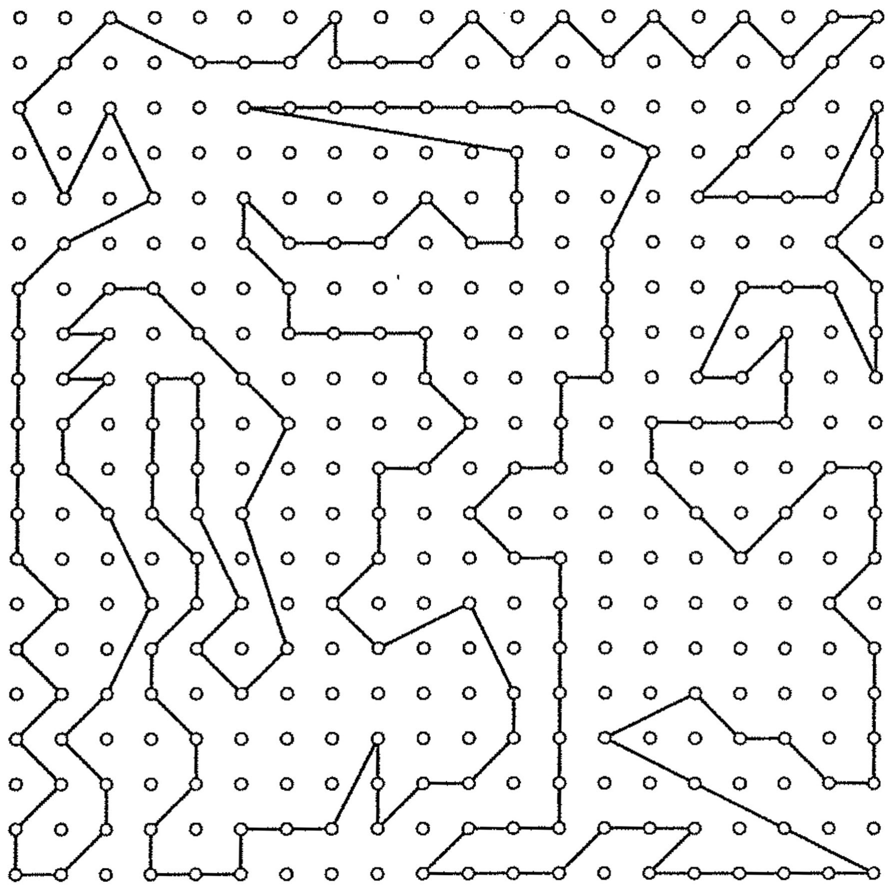
THE PROBLEM:

Suppose you have a square grid orchard, and you wanna graze your goats inside the following fence in your orchard. Each goat can graze exactly the amount of the grass contained in one square cell of the grid: *one square, one goat*. How many goats can graze?[1]

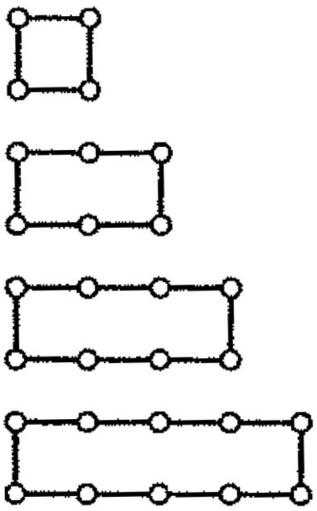


We are going to derive a formula for the area of a lattice polygon in terms of the number of lattice points inside and on its boundary, and then use this formula to solve our problem. Turn the page if you are interested!

[1] Our reference is Stewart, I., *Another Fine Math You've Got Me Into ...*, Dover Publications, 1992, Chapter 5. The idea of discussing this problem was suggested by Blake.

1. For a polygon P, let S be the area[2] of P, let I be the number of lattice points inside P, and let B be the number of lattice points on the boundary of P.

   For each of the following rectangles, what is S, I and B?

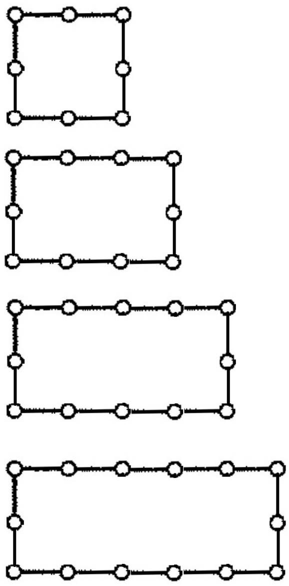


Rectangle	S	I	B
1×1			
1×2			
1×3			
1×4			

2. What is the pattern? (I mean, when $I = 0$, what is the relation between S and B?)

[2] By area, I mean the minimum number of unit square cells needed to cover polygon P completely.

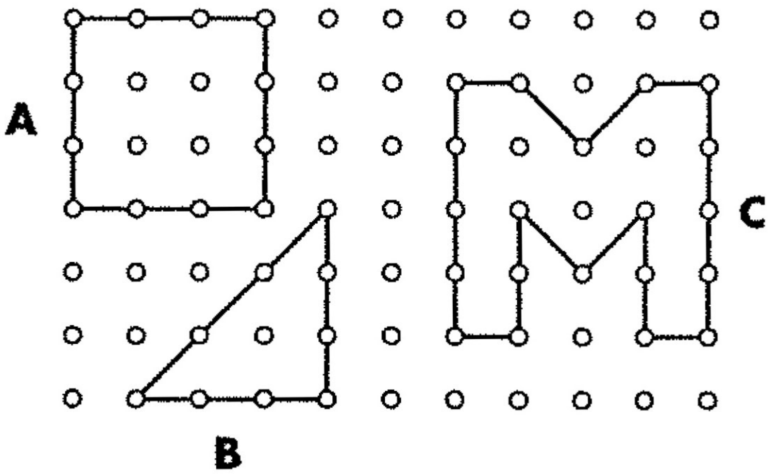
3. For the following rectangles, complete the table.



Rectangle	S	I	B	$\frac{B}{2}-1$	$S-\left(\frac{B}{2}-1\right)$
2×2					
2×3					
2×4					
2×5					

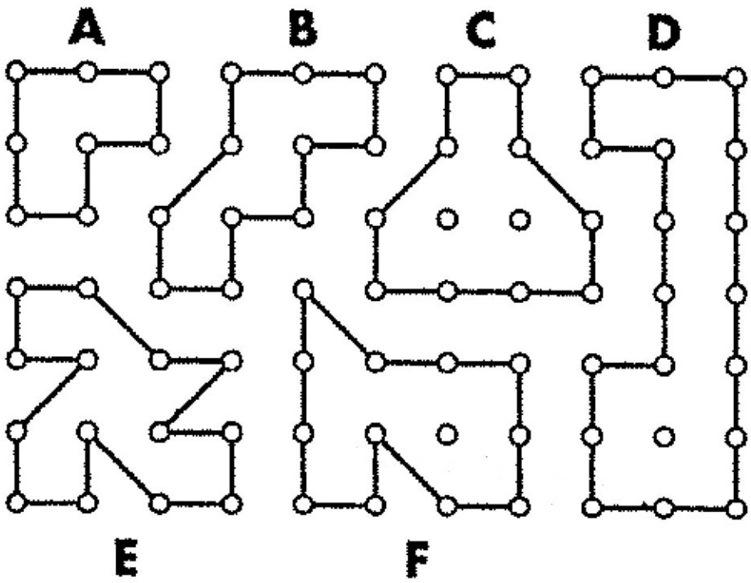
4. What is pattern? (I mean what is the relation bet between S, I and B?)

5. Test Pick's formula $S = I + \frac{B}{2} - 1$ for the following polygons.



Polygon	S	I	B	$I + \frac{B}{2} - 1$
A				
B				
C				

6. Test Pick's formula $S = I + \frac{B}{2} - 1$ for the following polygons.



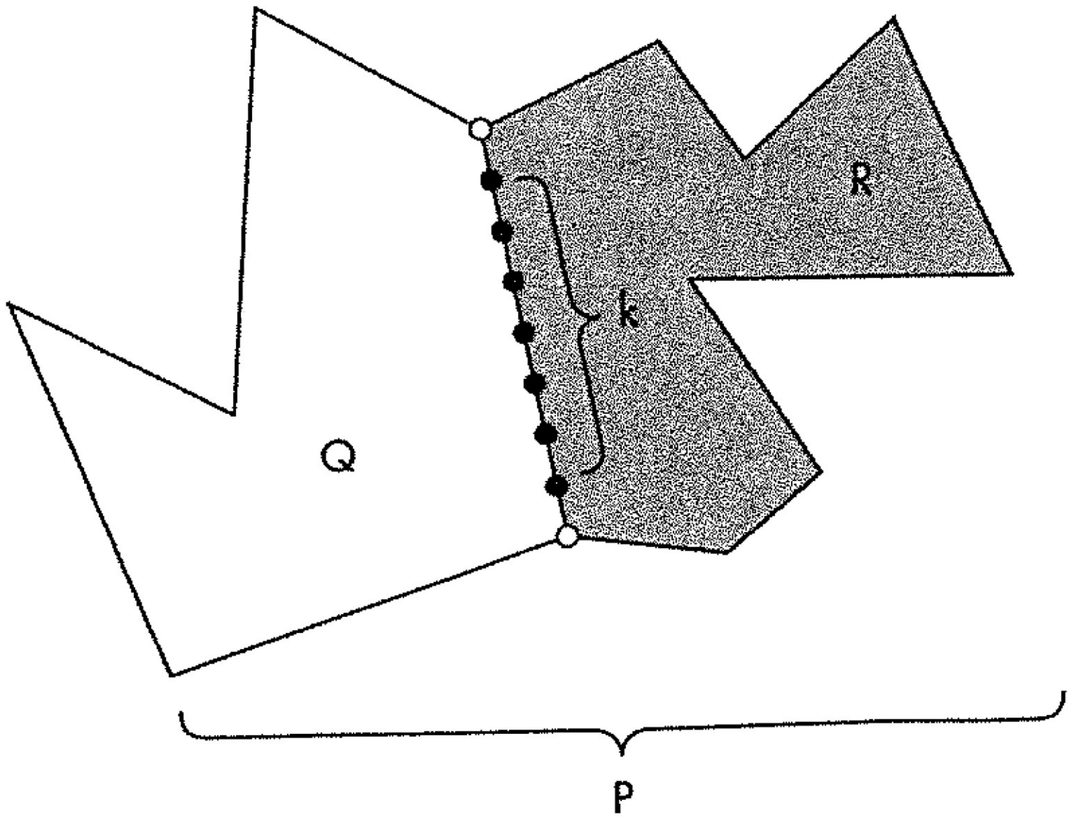
Polygon	S	I	B	$I + \frac{B}{2} - 1$
A				
B				
C				
D				
E				
F				

7. Prove Pick's foormula for a general right rectanle of length L and width W.

Polygon	S	I	B	$I+\frac{B}{2}-1$
Rectangle $L \times W$				

Now, we start to convice ourselves that the Pick's formula is valid for every lattice polygon. We do this in several steps.

STEP I. Consider lattice polygons Q and R, joined along a segment to form a polygon P, as depicted in the following figure.



8. What is I_P (the number of internal points of P) in terms of I_Q (the number of internal points of P), I_R (the number of internal points of R) and k?

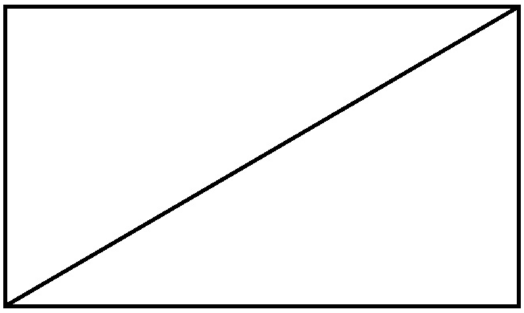
9. What is B_P (the number of boundary points of P) in terms of B_Q (the number of boundary points of Q), B_R (the number of boundary points of R) and k?

10. Show that

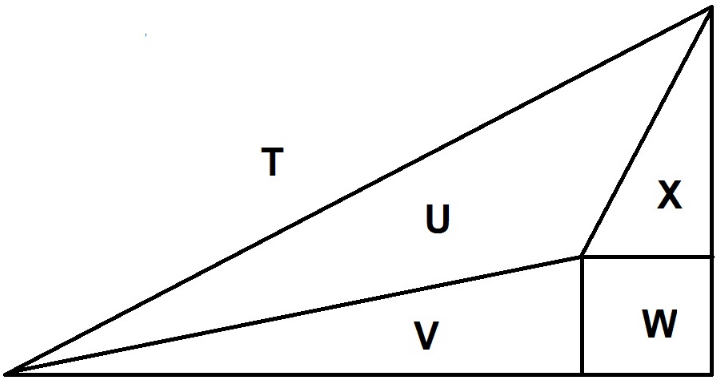
$$I_P + \frac{B_P}{2} - 1 = \left(I_Q + \frac{B_Q}{2} - 1\right) + \left(I_R + \frac{B_R}{2} - 1\right).$$

11. Show that, if Pick's formula is true for polygons Q and R, then it is true for P. (This is called the **additivity** of the Pick's quantity $I + \frac{B}{2} - 1$.)

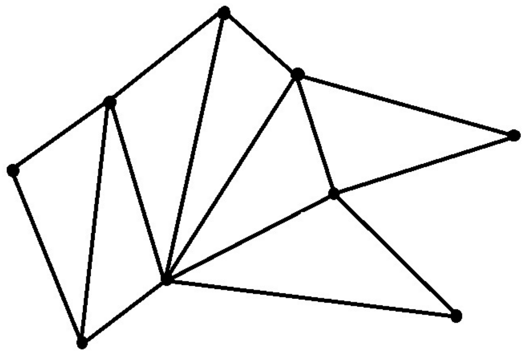
12. (*STEP II.*) We have already proved Pick's formula for right rectangles, in activity 7. Now prove it for right triangles. [Hint: use activity 11.]



13. (*STEP III.*) One can express any tringle (U in the figure below) in terms of right triangles (T, V, X) and rectangles (*W*). Use this observation, to prove Pick's formula for any U.



14. (*STEP IV.*) Every polygon can be cut up into triangles, as the following figure shows. Use this observation, to prove Pick's formula for any polygon.



15. Now we are sure about the validity of Pick's formula. Use this formula to calculate the number of goats can graze in your orchard, inside the fenced area shown below.