The Pigeonhole Principle

Theorem 1(Pigeonhole Principle). To put n+1 objects into n boxes, at least one box has to contain at least two objects.

Example. Among 27 people there are two who have the same capital letter in their last name.

Q1 Among how many people can you guarantee that two of them have the same initials. (only first name and last name considered. For example, Albert Einstein's initial is AE, and Steven Hawking's initial is SH)

Q2 A deck of cards contains 13 hearts, 13 spades, 13 diamonds, 13 clubs and 2 jokers. Say you are asked to draw cards randomly from a deck. At least how many cards do you have to draw to guarantee that you have 2 of the same kind? And at least how many cards guarantee you two spades?

Let X and Y be sets and f be a function from X to Y. That is, f maps every element in X to a unique element in Y. (It may map several elements in

X to the same element in Y) We call f one-to-one if or any two different elements a,b in X, f(a) is different from f(b) in Y. We call f onto if for any element c in Y, there is some element d in X such that f(d)=c.

Q4 Let X and Y be FINITE sets, show that if X has more elements than Y, then f is not one-to-one.

Q5 Let X and Y be FINITE sets with the same number of elements and suppose f is onto, show that f is one-to-one.

Q6 If X is allowed to be an INFINITE set, can you think of an example of a function f from X to X such that f is one-to-one but not onto? How about f from X to X being onto but not one-to-one?

Q7 Consider the integers 1,2,3,…,98,99,100. Using Pigeonhole Principle to show that for any 51 integers we choose (from 1 to 100), there are two such that one divides the other.

Theorem 2(Strong Pigeonhole Principle) Say q_1, q_2, q_3,…, q_n are positive integers. To put $q_1+q_2+q_3+\ldots+q_n-n+1$ objects into n boxes, then either the first box contains at least q_1 objects, or the second box contains at least q_2 objects, …, or the n^{th} box contains at least q_n objects. (why?)

Q8 If the average of k positive integers is greater than r, show that at least one of the integers is greater than r.

Q9 A store has 4 kinds of juice: apple, orange, watermelon and graph. At least how many bottles of juice do you have to buy such that you have either at least 7 bottles of apple juice, or at least 8 bottles of orange juice, or at least 10 bottles of watermelon juice, or at least 4 bottles of grape juice?

Now, let’s take a look at an interesting (really hard) case.

Q10(Ramsey) Show that in a group of 6 people, either there are three, each pair of whom know each other, or there are three, each pair of whom don't know each other.

Q10* Can you prove the previous problem for a group of more than 6 people?

More to think about…

1.Show that if n+1 integers are chosen from the set {1,2,3, … , 2n}, then there are always two which differ by 1.

2.Show that if 2n+1 integers are chosen from the set {1,2,3, … , 6n}, then there are always two which differ by at most 2.

Can you generalize the last two problems into a principle?

3.There are 50 soccer balls, 50 basket balls, 50 tennis balls and 50 base balls in the gym. Say every minute you get one ball out of the gym, how long does it take before you know for sure you have at least 20 balls of the same kind?

4.Use the pigeonhole principle to prove tat the decimal expansion of a ratio m/n, where m and n are both positive integers, is eventually repeating itself. For example,
16/73=0.2191780821917808219178082191 7808…… and
34478/99900=0.345125125125125125125125……

5.In a room there are 10 people, none of whom are older than 60 (consider only integer ages only) but each of whom is at least 1 year old. Show that one can always find two disjoint groups of people (so no one is in both groups) the sum of whose ages is the same. Can you replace 10 by a smaller number to maintain the same conclusion?

Just for fun…

Sun and Pang are two really smart guys in ancient China. One day, their teacher picked two integers from 2 to 99. He told Pang the sum of the two integers and told Sun the product of the two integers. But neither Sun or Pang knows the number of each other. The next day, Pang met Sun and told him that he couldn’t figure out what the two numbers are, but he knew for sure that Sun wouldn’t know either. ‘I didn’t know it in the first place, but now I do.’ said Sun. ‘Well in that case, I do too.’ said Pang.

What are the two integers?