"Dependence"

Math Circle

October 15, 2016

1 Warm up games

1. Flip a coin and take it if the side of coin facing the table is a "head". Otherwise, you will need to pay one.
   Will you play the game? Why?

2. If Alice flips the coin and tells you that it comes up a "tail", then it is your turn to flip. The rules is also take it if the side of coin facing the table is a "head" and pay one otherwise.
   Will you play the game? Why? What is the probability that you get a "head"?

3. Alice flips the coin and tells you that it comes up a "tail". The rules is also take it if the side of coin facing the table is a "head" and pay one otherwise.
   Will you play the game? Why? What is the probability that you get a "head"?

2 Some events are dependent on others

1. You flip two coins. The first coin shows a "head" and the other die rolls under the table and you cannot see it. Now, what is the probability that both coins show "head"?

2. You roll two dice. The first die shows a ONE and the other die rolls under the table and you cannot see it. Now, what is the probability that both die show ONE?

3. We want to draw one card from a standard deck of 52 cards. What is the probability that it is a king? If it is a king, and we draw another, what is the probability that the second one is also a king? Which one is larger? (Could you tell by intuition first, then calculate to check?)

4. What is the probability of drawing two aces from a standard deck of 52 cards, given that the first card is an ace? The cards are not returned to the deck. (Compare your result with problem 3.)

5. You decide to tell your fortune by drawing two cards from a standard deck of 52 cards. What is the probability of drawing two cards of the same suite in a row? The cards are not replaced in the deck.

6. Three people(A B C) line up to board an airplane (other people already sitted), but the first(A) has lost his boarding pass and takes a random seat instead. Each subsequent passenger takes his or her assigned seat if available, otherwise a random unoccupied seat.

   What is the probability that the last passenger(C) to board finds his seat occupied?

   (You can form a group of three to try yourself. Can C finally sit on B's seat?)

7. Advanced question: former question with four people instead of three. How about five, six? Could you guess the result for one hundred people?

3 Flipping odd coins

Suppose we have three coins. One of the coins is drawn at random and flipped; it comes up "heads". What is the probability that there is a head on the other side of this coin?

3.1 A two-tailed coin and two ordinary coin

3.2 A two-tailed coin, a two-headed coin and two ordinary coin

Is the answer the same as the first one?
How about try several times to verify your answer?

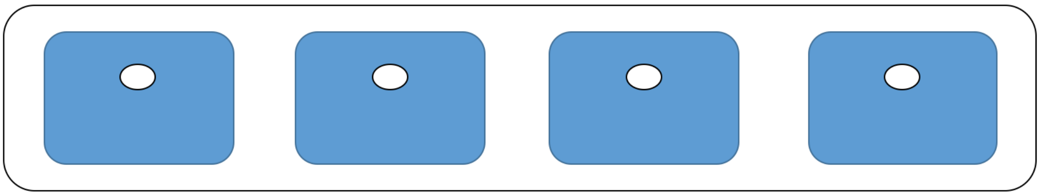
4 Numbers in Boxes

The grand opening of a candy bar offers a game to get candy.

The game includes 4 players, who are numbered from 1 to 4. A room contains a cupboard with 4 drawers. The numbers randomly are placed in these 4 drawers.

The players enter the room, one after another. Each prisoner may open and look into 2 drawers in any order. The drawers are closed again afterwards.

If, during this game, every player finds his number in one of the drawers, all players are getting a bag of candy. If just one player does not find his number, all players lose.



4.1 Choose randomly

1. If you are one of the players, and you randomly choose 2 boxes, what is the probability you find your own name?

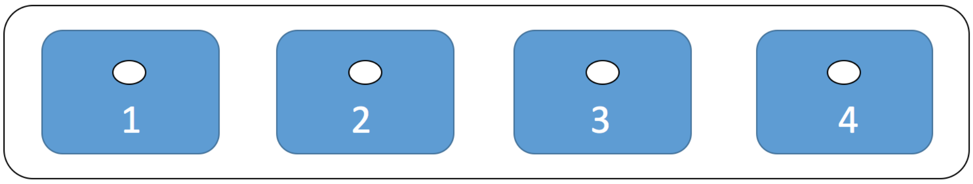
2. If you and one of your friends are going to play, and you both randomly choose boxes, what is the probability you both find your own name?

3. If you and two of your friends are going to play, and you all randomly choose boxes, what is the probability you all find your own name?

4. If you and three of your friends are going to play, and you all randomly choose boxes, what is the probability you all find your own name and win the candy?

4.2 Do you have any other strategy in order to get the candy?

4.3 Strategy: using the information gained from previously opened drawers

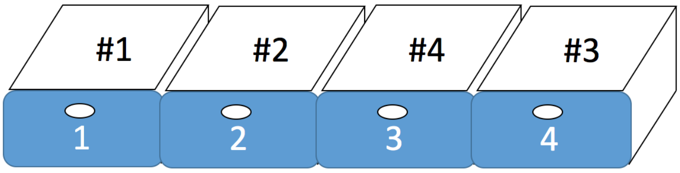


Numbered the drawers from 1 to 4, then

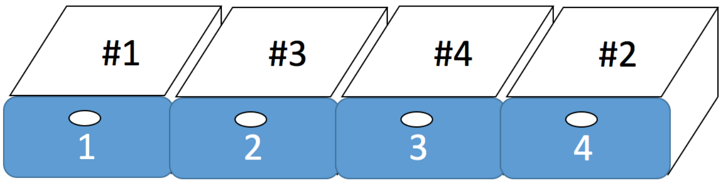
- Each player first opens the drawer with his own number.
- If this drawer contains his number he is done and was successful.
- Otherwise, the drawer contains the number of another player and he next opens the drawer with this number.

Is this a good strategy?

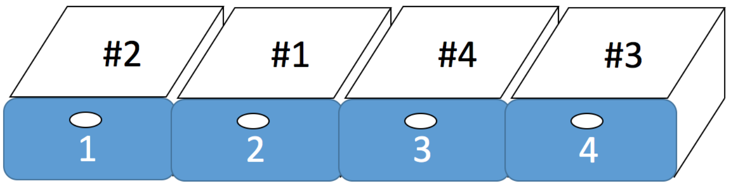
1. If #1 is in drawer 1, #2 is in drawer 2, #3 is in drawer 4 and #4 is in drawer 3, does this strategy work?



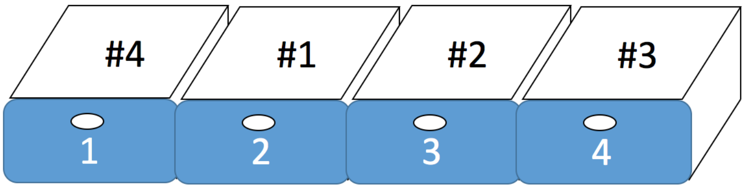
2. If #1 is in drawer 1, #3 is in drawer 2, #4 is in drawer 3 and #2 is in drawer 4, does this strategy work?



3. If #1 is in drawer 2, #2 is in drawer 1, #3 is in drawer 4 and #4 is in drawer 3, does this strategy work?



4. If #4 is in drawer 1, #1 is in drawer 2, #2 is in drawer 3 and #3 is in drawer 4, does this strategy work?



5. What about the other situations? Does the strategy work in that situation?

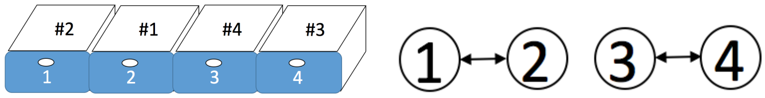
6. What is the probability to win the candy? Is it the same as before(randomly choose)?

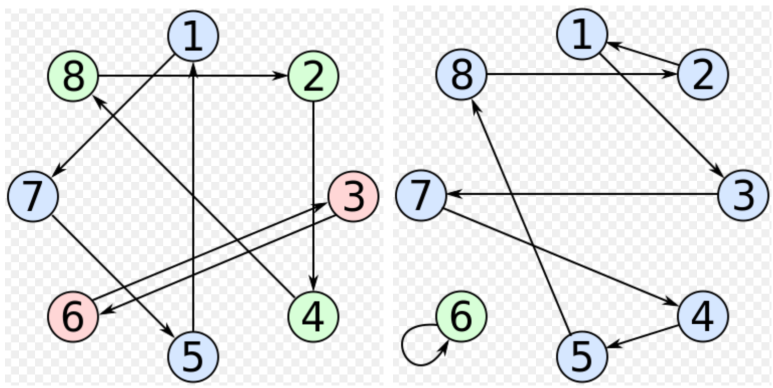
☺Because the success of one player is not independent to another.

Is this a good strategy?

4.4 Advanced question: what will happen if increasing the number player to 8?

Use **Graph representations** of the permutations





4.5 "Advanced-advanced" question: what will happen if increasing the number player to 100?

The players will lose if there exist a loop consist of more than 50 nodes in the graph representation (called cycle of length l, $l > 50$).

A permutation of the numbers 1 to 100 can contain at most one cycle of length $l > 50$. There are exactly $\binom{100}{l}$ ways to select the numbers of such a cycle.

Within this cycle, these numbers can be arranged in $(l-1)!$ ways since there are $l-1$ possibilities to select the starting number of the cycle. The remaining numbers can be arranged in $(100-l)!$ ways. Therefore, the number of permutations of the numbers 1 to 100 with a cycle of length $l > 50$ is equal to $\binom{100}{l} \cdot (l-1)! \cdot (100-l)! = \frac{100!}{l}$.

The probability, that a (uniformly distributed) random permutation contains no cycle of length greater than 50 is with the formula for single events and the formula for complementary events thus given by

$$1 - \frac{1}{100!}\left(\frac{100!}{51} + \ldots + \frac{100!}{100}\right) = 1 - \left(\frac{1}{51} + \ldots + \frac{1}{100}\right) \approx 0.31183$$

5 Reference

https://en.wikipedia.org/wiki/100_prisoners_problem