We are going to learn something about the most important mathematical constant

$$\pi$$

which is the ratio of the circumference of a circle to its diameter. It is approximately

$$\pi \approx 3.14159265359.$$

In more decimals

$$\pi \approx 3.1415926535897932384626433832795028841971693993751 0.$$

Here are the first one thousand digits of π.

3.1415926535 8979323846 2643383279 5028841971 6939937510 5820974944
5923078164 0628620899 8628034825 3421170679 8214808651 3282306647 0938446095
5058223172 5359408128 4811174502 8410270193 8521105559 6446229489 5493038196
4428810975 6659334461 2847564823 3786783165 2712019091 4564856692 3460348610
4543266482 1339360726 0249141273 7245870066 0631558817 4881520920 9628292540
9171536436 7892590360 0113305305 4882046652 1384146951 9415116094 3305727036
5759591953 0921861173 8193261179 3105118548 0744623799 6274956735 1885752724
8912279381 8301194912 9833673362 4406566430 8602139494 6395224737 1907021798
6094370277 0539217176 2931767523 8467481846 7669405132 0005681271 4526356082
7785771342 7577896091 7363717872 1468440901 2249534301 4654958537 1050792279
6892589235 4201995611 2129021960 8640344181 5981362977 4771309960 5187072113
4999999837 2978049951 0597317328 1609631859 5024459455 3469083026 4252230825
3344685035 2619311881 7101000313 7838752886 5875332083 8142061717 7669147303
5982534904 2875546873 1159562863 8823537875 9375195778 1857780532 1712268066
1300192787 6611195909 2164201989

1. Does your calculator or cellphone compute π? It is OK if not.

2. Complete the following table, relating the circumference of a circle to the length of its diameter.

diameter (in meters)	circumference (in meters)
1	
2	
5	

3. π is also the ratio of the area of a circle to the square of its radius. Complete the following table, relating the area of a circle to the length of its radius.

radius (in meters)	area (in square meters)
1	
2	
	54

4. Archimedes computed π by approximating the circumference of circle by inscribed and circumscribed regular polygons. Here is his method.

Start with initial values

$$C_1 = 4, \quad I_1 = 2\sqrt{2},$$

and let for each $n \geqslant 1$

$$C_{n+1} = \frac{2C_n I_n}{C_n + I_n}, \qquad I_{n+1} = \sqrt{C_{n+1} I_n}.$$

For example

$$C_2 = \frac{2C_1 I_1}{C_1 + I_1}, \quad I_2 = \sqrt{C_2 I_1},$$

$$C_3 = \frac{2C_2 I_2}{C_2 + I_2}, \quad I_3 = \sqrt{C_3 I_2},$$

$$C_4 = \frac{2C_3 I_3}{C_3 + I_3}, \quad I_4 = \sqrt{C_4 I_3}.$$

Complete the following table, using your calculator.

n	C_n	I_n
1		
2		
3		
4		
5		

Are you getting better and better approximations for π through numbers $C_1, C_2, C_3, \ldots$?

What about $I_1, I_2, I_3, \ldots$?

5. Ramanujan, an Indian mathematicien, found the following elegant formula for computing π.

$$\pi = \frac{\frac{9801}{2\sqrt{2}}}{\sum_{n=0}^{\infty} \frac{(4n)!\ (1103+26390n)}{(n!)^4 (396)^{4n}}}.$$

By this I mean π can be approximated successively by the following numbers which I ask you to compute using your calculator.

$$\pi \approx \frac{\frac{9801}{2\sqrt{2}}}{1103} =$$

$$\pi \approx \frac{\frac{9801}{2\sqrt{2}}}{1103 + \frac{(1\times2\times3\times4)\times(1103+26390)}{1^4\times396^4}} =$$

$$\pi \approx \frac{\frac{9801}{2\sqrt{2}}}{1103 + \frac{(1\times2\times3\times4)\times(1103+26390)}{1^4\times396^4} + \frac{(1\times2\times3\times\cdots\times8)\times(1103+26390\times2)}{(1\times2)^4\times396^8}} =$$

Are you getting better and better approximations for π?

6. Lord Brouncker found the following approximation for π by *continued fractions*.

$$\pi = \cfrac{4}{1+\frac{1^2}{3+\frac{2^2}{5+\frac{3^2}{7+\cdots}}}}$$

By this I mean π can be approximate successively by the following numbers which I ask you to compute using your calculator.

$$\pi \approx \frac{4}{1+\frac{1^2}{3}} =$$

$$\pi \approx \frac{4}{1+\frac{1^2}{3+\frac{2^2}{5}}} =$$

$$\pi \approx \frac{4}{1+\frac{1^2}{3+\frac{2^2}{5+\frac{3^2}{7}}}} =$$

Are you getting better and better approximations for π?

What is the next approximation in this method? Compute it with your calculator.

Congradulations for completing the previous activities! The next two activities are for those of you who know some very elementary triginometry. It justifies the Archimedes method (used in activity 4) to approxomate π.

Fix a circle of unit diameter, and let $n \geqslant 3$ be an arbitrary natural number. The idea is to approximate the circumference of this circle (which equals π) by the circumference of regular n sided polygons inscribed and circumscribed in our circle.

7. Let C and I, respectively, denote the circumference of the circumscribed and inscribed regular n-gons in our circle. Can you find a formula for C and I, relating them to n?

   (Hint: *concentrate on one edge of your polygon; use trigonometric functions* sin *and* tan; maybe it helps to look at the answer in the footnote[1].)

[1] Answer: $C = n \tan \frac{\pi}{n}$, and $I = n \sin \frac{\pi}{n}$.

8. Let C' and I', respectively, denote the circumference of the circumscribed and inscribed regular $2n$-gons in our circle. By the previous activity you know that

$$C = n \tan \frac{\pi}{n}, \quad I = n \sin \frac{\pi}{n}, \quad C' = 2n \tan \frac{\pi}{2n}, \quad I' = 2n \sin \frac{\pi}{2n}.$$

Can you prove the follwoing two formulas?

$$C' = \frac{2CI}{C + I}, \quad I' = \sqrt{C'I}.$$

(Hint: *for the first one, start from* $\frac{2CI}{C+I} = \frac{2}{\frac{1}{C}+\frac{1}{I}}$*, and use trigonometric identity* $\frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}$*; for the second one, start from* $\sqrt{C'I}$ *and use trigonometric identity* $\sin(2\theta) = 2\sin\theta \cdot \cos\theta$.)