DICE GAMES

WASHINGTON UNIVERSITY MATH CIRCLE --- FEBRUARY 12, 2017

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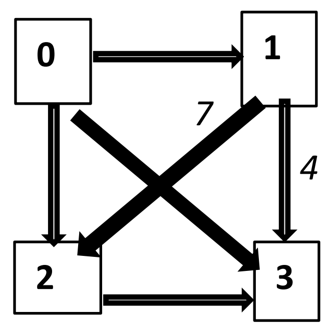
BRADLY EFRON DICE – *WHICH IS THE BEST DIE FOR WINNING THE GAME?*

I. DATA COLLECTION

This is a two-person game. Get into groups of 2 (or 3 if you wish). I will call the two players A and B. Each group gets one set of four different dice. The players may briefly study the dice.

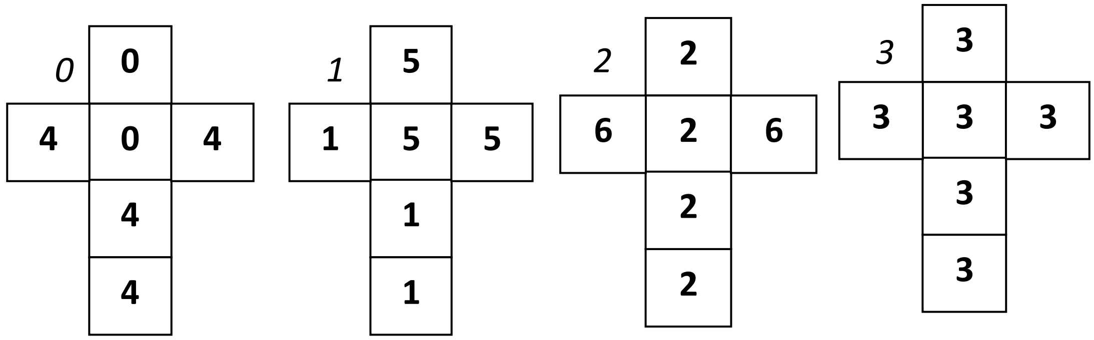
- Person A selects one die. Then B selects a different die.
- Each person rolls their die, higher number wins. There will be no ties in this game.
- With the SAME dice, A and B roll a total of 10 times and keep a record of wins and losses.
- To report your result to me, identify each die by its lowest number: 0, 1, 2, or 3. To avoid confusion, always start with the lower number. For example, ***"1 beat 3 four times"*** OR ***"1 beat 2 seven times"***
- Record your result on this diagram.
- Repeat one or two times, changing who selects a die first.



II. DISCUSSION: *Which die is "best"?*

That is, when Person A selects first, which die should A select? Then does it matter which die B selects?

Explain your answer.



III. **<u>Probability Analysis</u>** <u>TOGETHER</u>, we will compute the probability that ***Die "1" beats Die "2"*** in **<u>THREE ways</u>**.

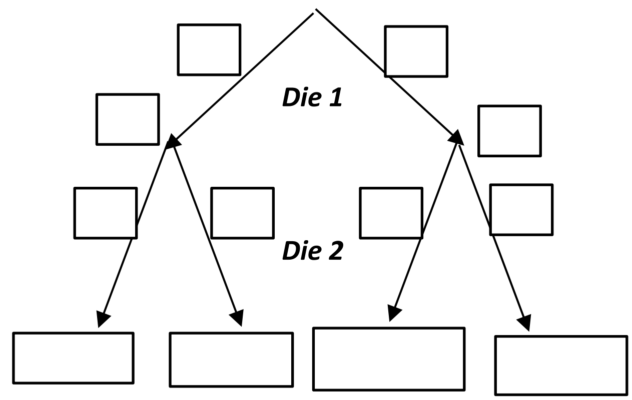
<u>TABLE</u>

W = *Die 1* beats *Die 2*						
	2	2	2	2	6	6
1						
1						
1						
5						
5						
5						

Complete the diagram. Then complete:

The probability that Die 1 beats Die 2 is ______.

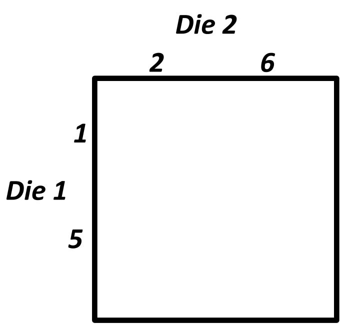
<u>TREE</u>



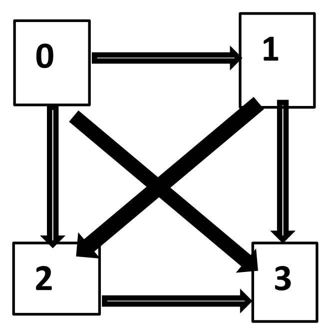
Complete the diagram. Then complete:

The probability that Die 1 beats Die 2 is ______.

<u>AREA</u>



<u>SUMMARY</u>



The probability that Die 1 beats Die 2 is ______.

<u>In your groups, compute the other five probabilities.</u>

Which die is "best"?* That is, if Person A selects first, which die should A select? *WHY?

If A selects Die 0, B should select Die _____. B wins with probability _____

If A selects Die 1, B should select Die _____. B wins with probability _____

If A selects Die 2, B should select Die _____. B wins with probability _____

If A selects Die 3, B should select Die _____. B wins with probability _____

<u>SUMMARY</u> -- This set of *Non-Transitive Dice* was designed in the early 1970's by Professor Bradley Efron of Stanford University for his introductory statistics class. Martin Gardner popularized these dice in his *Mathematical Games* column in the October 1974 issue of *Scientific American.*

EXTENSIONS

SUM OF TWO DICE Play a different two-person game with the same four dice. Again, Player A selects a die, then player B selects a die. This time each player rolls his/her *die TWICE and adds the two numbers.* Higher total wins. In case of a tie, ignore it and roll again. [Ties will make the probability analysis harder.]

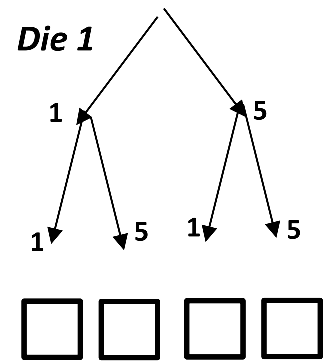
1. **Play 2 or 3 sets of 10 rounds of the new game**. Record your results and tell them to me.

2. **DISCUSS** --- ***Is there a BEST DIE in this game?***

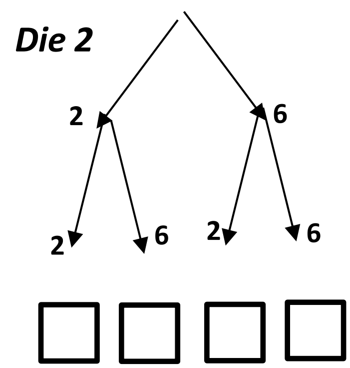
3. **PROBABILITY ANALYSIS** Compute the probability that ***Die 1*** beats ***Die 2.***

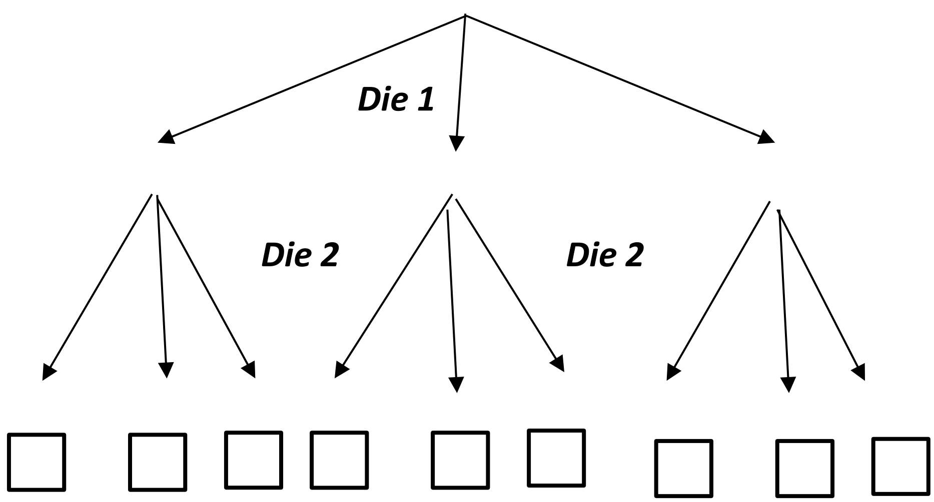
This is a *3-step* process.

- When **Die 1** is rolled twice, the sum could be ________; ______; or ________.
- With probabilities of: _______; _______; or _______.



- When **Die 2** is rolled twice, the sum could be ________; ______; or ________.
- With probabilities of: _______; _______; or _______.





When each die is rolled twice, the probability that Die 1 beats Die 2 is: __________.

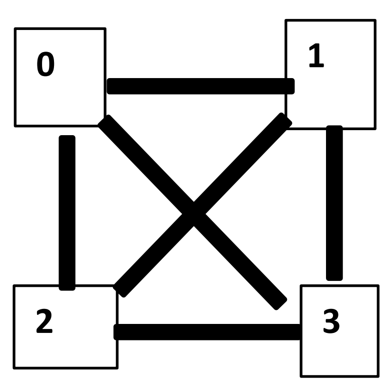
Compute one or two more probabilities.

Draw each segment as an arrow so that **each probability is at least 1/2.**

In the event of a tie, they roll the dice again.

Which is the BEST die? _________

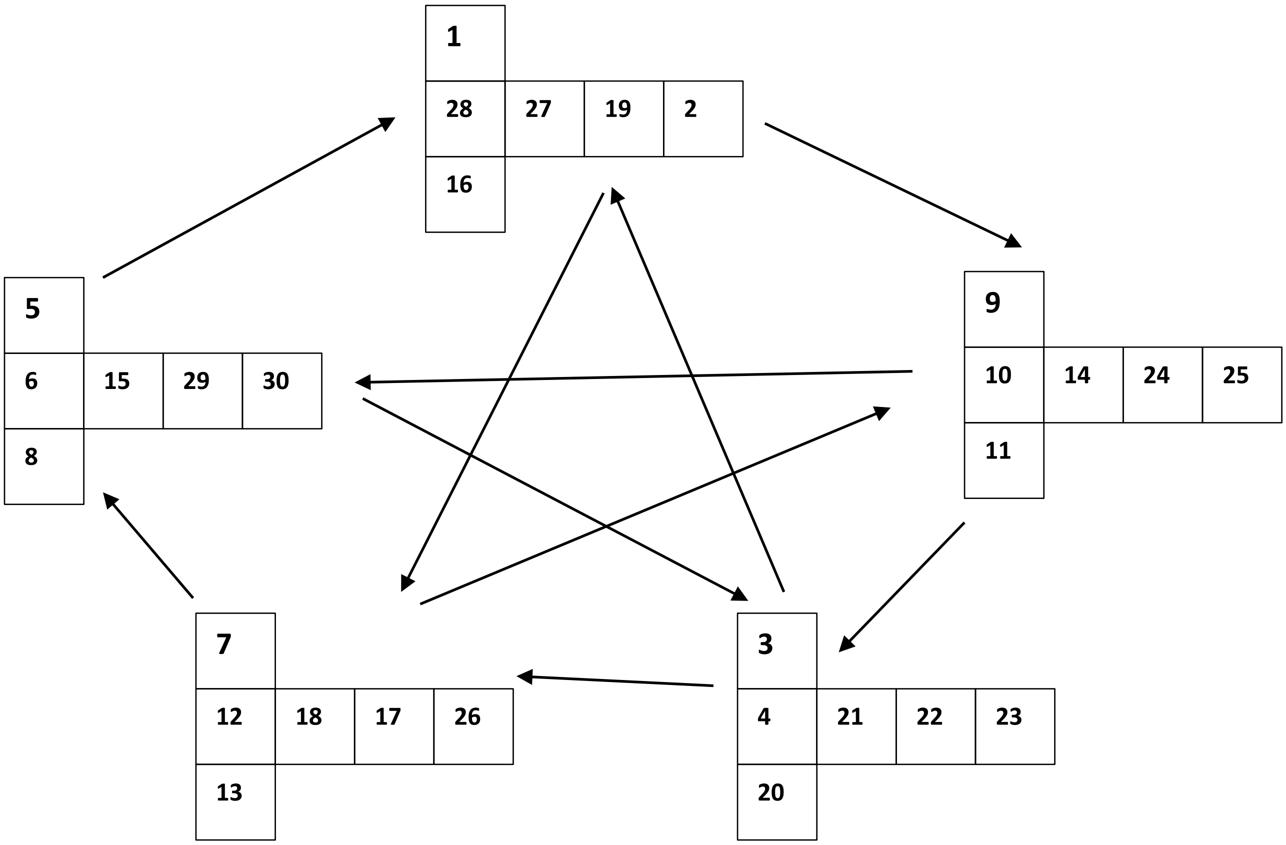
Which is the WORST die? ______



SILVEIRA'S SET OF 5 NON-TRANSITIVE DICE http://www.cut-the-knot.org/Probability/NonTransitiveDice.shtml

In 2016, Bráulio de O. Silveira from Brazil, created 7 different sets of 5 non-transitive dice. Here is one set.

Note that the 5 dice are labeled with the digits 1 through 30 without repetition. Again, identify each die by its least number. Note: On THIS diagram, the arrows are already pointed so that the probability on each arrow is at least 1/2.



1. From one of the five edges of the pentagon, select one pair of dice. Compute the probability for your pair. Repeat with one more pair of dice which form an edge. Report your answers to me.

HINT: *When a die has consecutive numbers, they can be treated as equal. How does this help your calculations?*

PAIR? _____________PROB?_______________; PAIR? ___________PROB? ___________

2. On each die, what is the sum of its six numbers? ___________

3. From the diagram, find all sets of 5 non-transitive dice: _______________________________

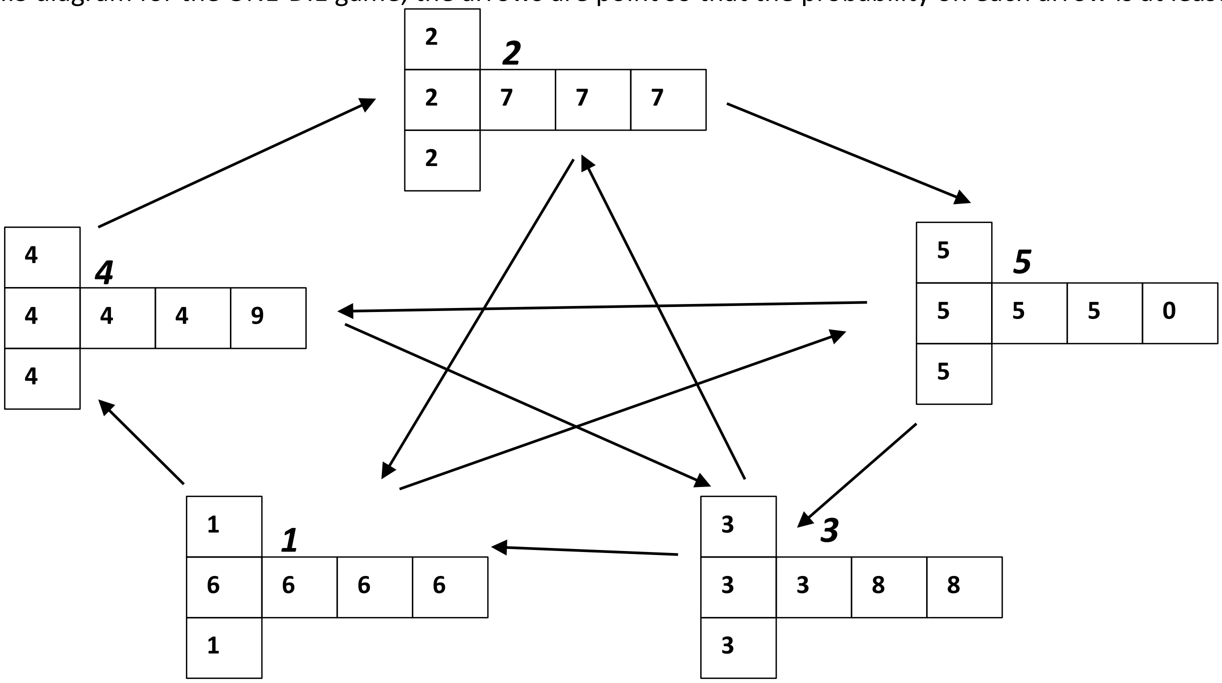
4. From the diagram, find all sets of 4 non-transitive dice: _______________________________

5. From the diagram, find all sets of 3 non-transitive dice: _______________________________

GRIME'S SET OF 5 NON-TRANSITIVE DICE

http://singingbanana.com/dice/article.htm .

On THIS diagram for the ONE-DIE game, the arrows are point so that the probability on each arrow is at least 1/2.



1. From the diagram: A. find TWO sets of 5 non-transitive dice; B. find sets of 4 non-transitive dice; and C. find sets of 3 non-transitive dice.

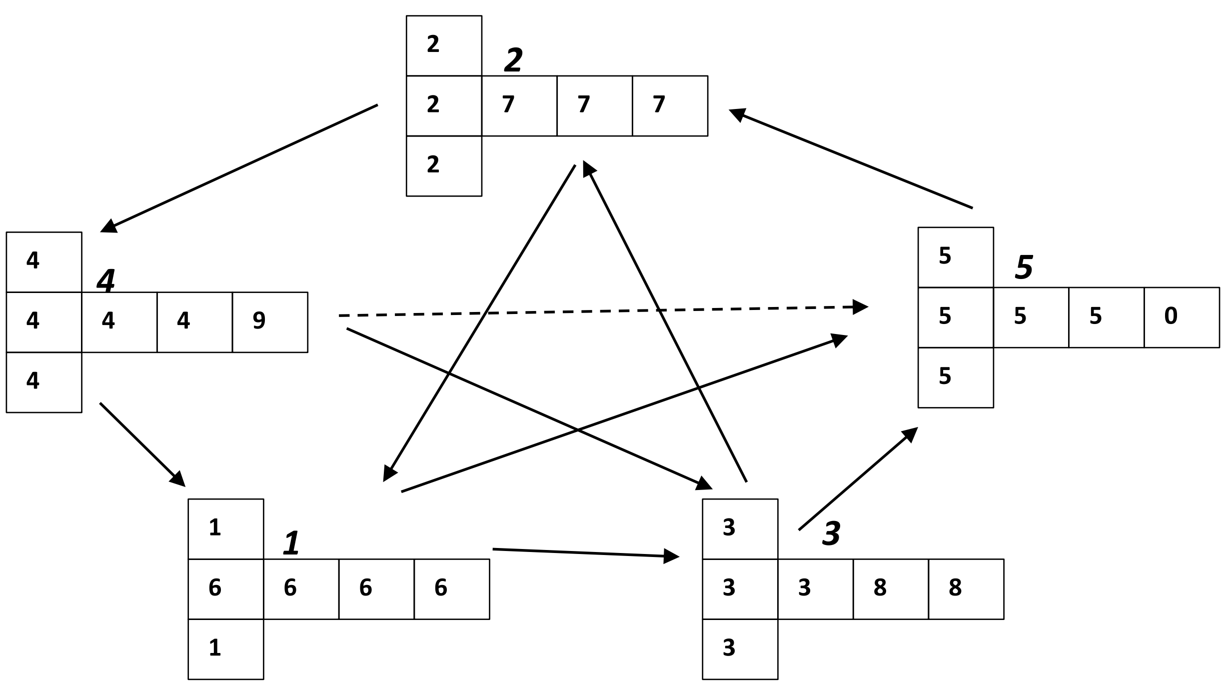
2. Compute the probability of ***Die 2 beats Die 5*** ___________ and the probability of ***Die 5 beats Die 4*** __________.

The following diagram represents the probability results for the ***sum of two-dice game*** with Grime's Dice. Again, each arrow is pointed to represent a probability greater than 1/2.

3. Compare the two diagrams. With one exception, what do you notice?

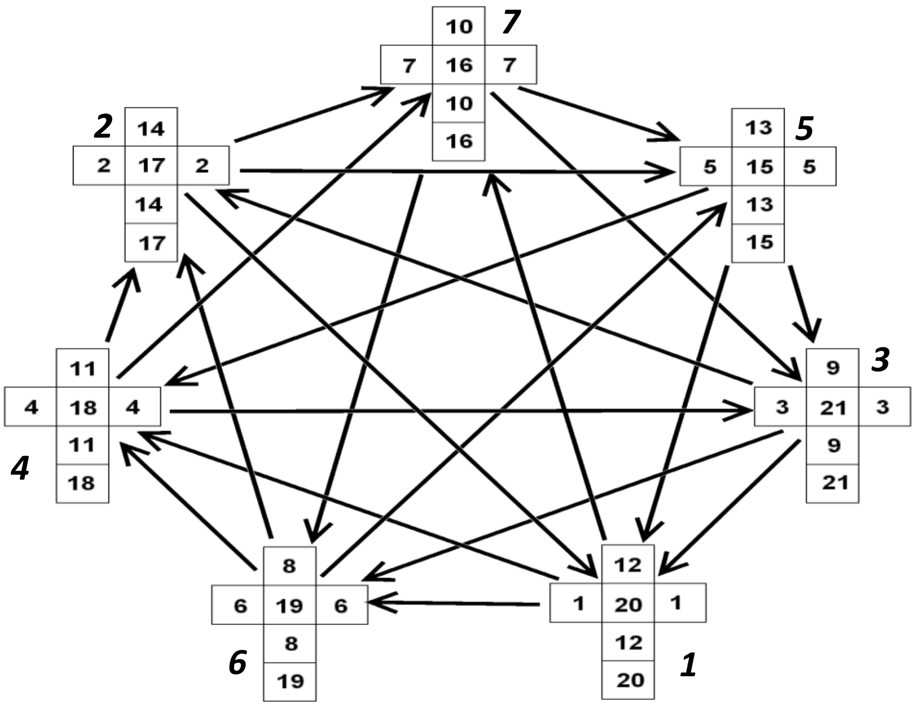
Note: After Grime created this set, someone found an error. Let's calculate that probability.

4. For the **sum of two dice**, compute the prob of ***Die 2 beats Die 5*** ______ and the prob of ***Die 5 beats Die 4*** ________.



DEVENTER'S SET OF 7 NON-TRANSITIVE DICE

https://en.wikipedia.org/wiki/Nontransitive_dice#cite_note-3



Each arrow indicates which die beats which die in the *one-die game*. Note that the 7 arrows on the edge are all clockwise. Therefore these 7 dice are non-transitive.

1. Can you find a different non-transitive sequence of all 7 dice? ____________

2. From this set, find at least one set of N dice that are non-transitive for:

N=6:_________________; N=5: __________________; N=4: _________________; N=3: _____________

3. How many arrows are in the diagram? ________

4. For this set of 7 dice, are the sums of the six numbers on each die the same? ____________

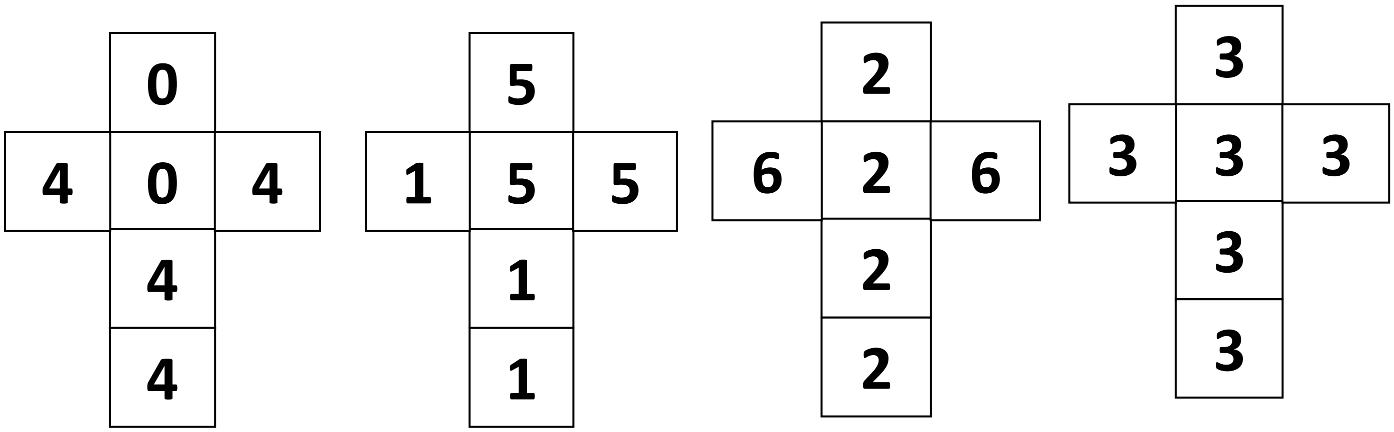
5. Amazingly, all of these arrows represent the SAME probability. As a way to check yourself, compute two of those probabilities.

6A. Two of your friends select Dice #6 and #7. Which die can you then select to beat EACH of them [two 2-person games]?

6B. Two of your friends select Dice #2 and #5. Which die can you then select to beat EACH of them [two 2-person games]?

6C. For ANY pair of dice your two friends select, you can always select one which will beat EACH of them. What is the probability that you will beat BOTH of them?

MISCELLANEOUS CHALLENGES **Efron's Dice**



Some people would prefer that Efron's dice did not have so many duplicate numbers.

Create a set of 4 dice labeled 1 through 24 that are equivalent to Efron's dice.

Die 0: ___; ___; ___; ___; ___; ___ Die 1: ___; ___; ___; ___; ___; ___

Die 2: ___; ___; ___; ___; ___; ___ Die 3: ___; ___; ___; ___; ___; ___

Hint: In assigning numbers from the list 1-24, consider the end numbers first.

HEATH DICE (1927)

In 1927, Royal Heath created a set of 5 dice. Each die was labeled with six distinct 3-digit numbers. When the 5 dice were rolled, Heath could compute their sum in about 5 seconds. Here are his dice:

A: 459; 558; 855; 657; 954; 7__ **B: 741; 543; 345; 840; 147; 6__**

C: 285; 384; 681; 483; 186; 7__ **D: 872; 773; 179; 278; 971; 3___**

E: 960; 663; 564; 267; 366; 1___

1: OOPS – two digits are missing from one number on each die – find them!

2 – What was Heath's method for adding the 5 numbers so quickly?

3 – What are the least and the greatest sums possible with these 5 dice?