WASHINGTON UNIVERSITY MATH CIRCLE

FINANCIAL MATHEMATICS
FROM GAMES OF CHANCE TO NO FREE LUNCH

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1. What “Financial Mathematics” do?

Contrary to common believe, a great deal of problems in financial mathematics has to do more with finding ways to avoid “free lunchs” (that is, making money for free) than to make someone “rich”. Financial mathematics has its origins in the problem of determining the “fair” price to pay for playing a game of chance (i.e., a game whose reward depend on an unpredictable event such as tossing a coin). However, financial mathematics shows that the “fair” price now of a “risky asset” (one whose future value is unpredictable) may be at odds with its “no free lunch” price in a free market. In this presentation, we will illustrate this using simple examples.

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2. Game of Chance

Activity 1: Consider the following games:

A) I toss a 1 dollar coin. If it lands on "heads", I will give you 2 dollars. If it lands on "tails", I will give you 50 cents.

B) Suppose that I toss the coin a second time. If it lands on "heads" again, I will double your earnings, while if it lands on "tails", I will take away half of your earnings.

C) [1] Suppose I repeat this $n-1$ additional times (in total I toss the coin n times). Each time, I will double your earnings if the coin lands on "heads", but I will take away half of those if the coin lands on "tails".

What should the corresponding fair prices be for playing each of the previous games? In other words, how much would you be willing to pay for playing each of the games?

[1] Challenging

Activity 2: Martingale betting strategy [2]:

Consider a game in which the gambler wins his stake if a coin comes up heads and loses it if the coin comes up tails. The “martingale strategy” is one in which the gambler double his bet after every loss and stops playing the first time he wins.

Suppose his original stake (how much he bets the first time) is $ 1 dollar.

A) **What would the gambler’s profit be if the coin comes up tails for the first time at the second toss?**

B) **What would the gambler’s profit be if the coin comes up tails for the first time at the third toss?**

C) **Can you generalize? Is there anything counterintuitive with your answer?**

Activity 3: [2] St. Petersburg paradox [1]:

A casino offers a game of chance for a single player in which a fair coin is tossed at each stage. The initial stake starts at 2 dollars and is doubled every time heads appears. The first time tails appears, the game ends and the player wins whatever is in the pot. Thus the player wins 2 dollars if tails appears on the first toss, 4 dollars if heads appears on the first toss and tails on the second, 8 dollars if heads appears on the first two tosses and tails on the third, and so on. Mathematically, the player wins 2^k dollars, where k equals number of tosses (k must be a whole number and greater than zero).

What would be a fair price to pay the casino for entering the game?

[2] Challenging

3. Options

Broadly, an option is a type of asset or investment whose final value or profit depends on the price of other asset. Let me illustrate this with an example:

Activity 4: Fair price of a put option [3]

Consider a large box of "coupons" whose value now is $ 1 dollar. At the end of my talk, I will toss a coin and the value of each coupon would be either 2 dollars or 50 cents depending on whether my coin lands on heads or tails.

I am aware that people are afraid to loose money (risk adverse) so I also create an "insurance" (call it put option), which will pay 50 cents per coupon if my coin lands on tails (so recovering the 50 cents someone will loose in his/her coupon) or nothing if my coin lands on heads. This insurance can be bought at some price, called the premium.

What should be the "fair" premium for having such an insurance?

Activity 5: No free lunch price of a put option.

Suppose I decide to sell the put option at \$ 0.25.

You are allowed to buy or sell any fraction of the coupon. So, 1/2 a coupon would be worth either \$1 or \$ 0.25 dollars depending on whether the coin lands on heads or tails.

We also assume that our faculty advisor, Blake, would serve as a bank in that he can lend any amount of money to anybody.

I claim that the \$0.25 premium would generate free lunch opportunities. Find a way to make money for free.

Activity 6: Can we generalize it?

Suppose that I will toss the coin three times at 1/2 hr intervals. Each time, I will double the value of each coupon if it lands on heads or half it if it lands on tails. So, for instance, the value per coupon would be \$ 8 dollars if I get heads every time, or \$ 2 if I get two heads and one tail, or \$ 0.5 if I get 1 heads and two tails, or \$ 0.125 if I get three tails.

I have again an insurance that will recover your loss in the value of a coupon if needed. So, the insurance will pay \$0.5 if I get two tails and 1 head or \$ 0.875 if I get three tails, and pay nothing in any other case.

What is the fair premium of the "insurance"?

Activity 7: Dynamic programming [4].

Suppose that we are allowed to buy and sell any fraction of the coupon and borrow any amount of money at time 0 and right after the first and second coin are tossed (at times 1/2 and 1).

[3] **Can you determine the no free lunch premium of the option?** i.e., how much should I charge so that there is no opportunities to make money for free.

[3]Challenging. Think backward in time.

References

[1] https://en.wikipedia.org/wiki/St._Petersburg_paradox.

[2] https://en.wikipedia.org/wiki/Martingale_(betting_system).

[3] https://en.wikipedia.org/wiki/Put_option

[4] https://en.wikipedia.org/wiki/Dynamic_programming