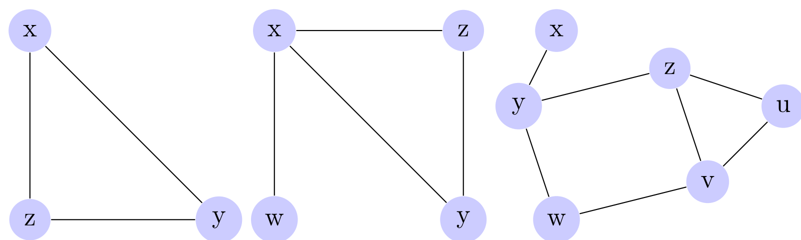
Graph Puzzles

October 1st, 2017

Today we are going to do some graph puzzles. A graph consists of “vertices” and connecting “edges”. Here are some examples:



1 Eulerian Path

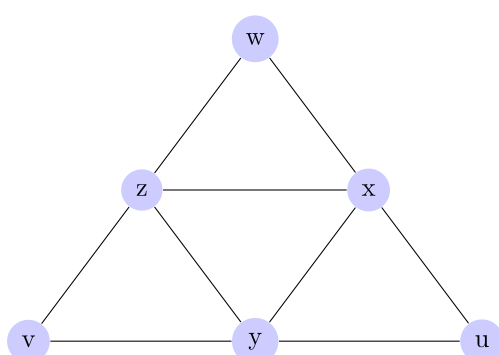
Given a graph, we would like to find a path with the following conditions:

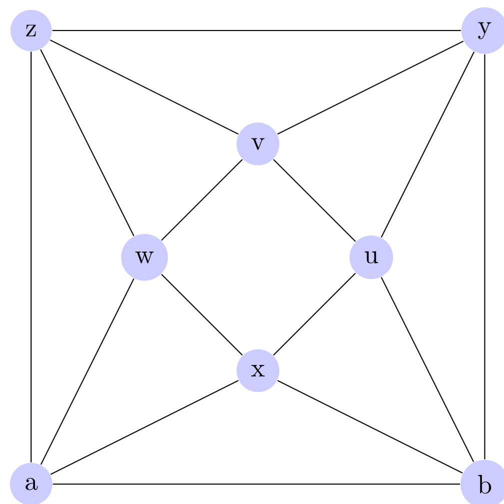
- the path should begin and end at the same vertex.
- the path should visit every edge exactly once.

In mathematics, such a path in a graph is called an **Eulerian path**. If a graph has an Eulerian path, then we say this graph is **Eulerian**.

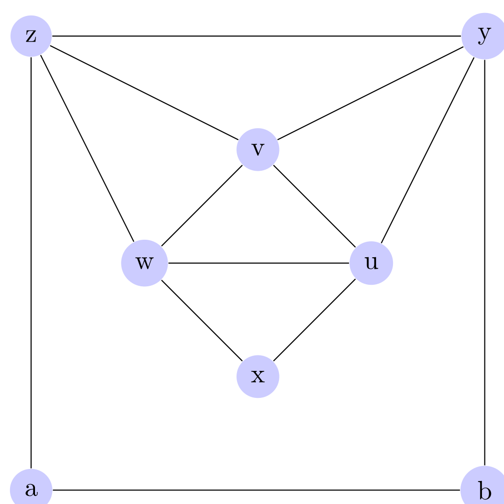
1. Find an Eulerian path in each of the following graphs:

(a)

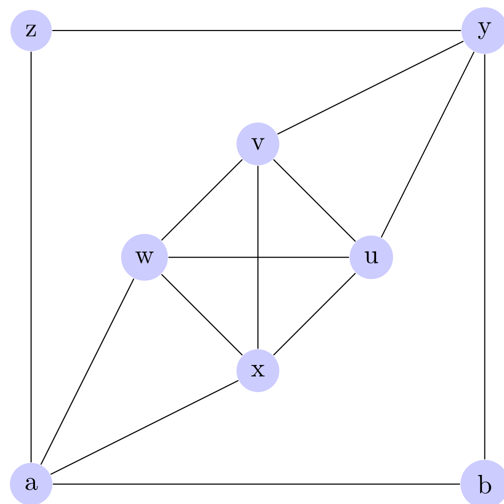




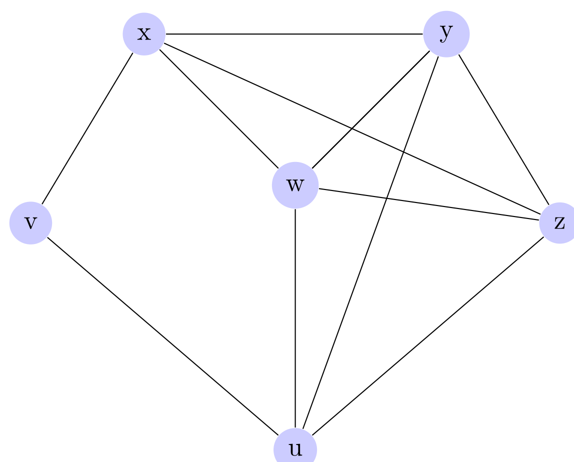
(b)



(c)



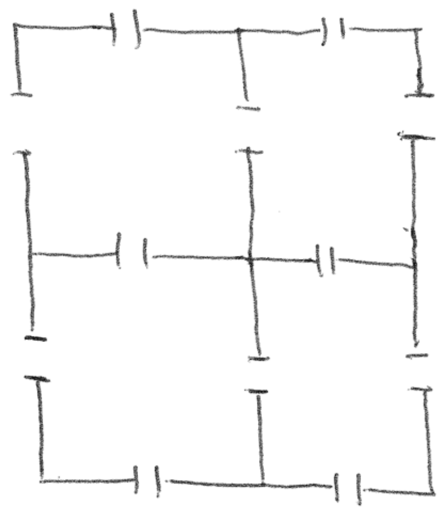
(d)



(e)

2 Room Puzzles

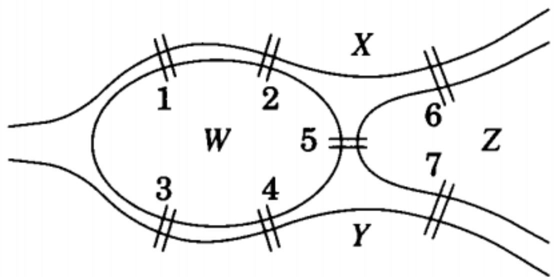
1. Consider the following four room apartment with doors.



(a) Can you find a continuous line that passes through each door exactly once?

(b) If we transform this floor plan into a graph, what should the vertices and the edges represent? What does the graph look like?

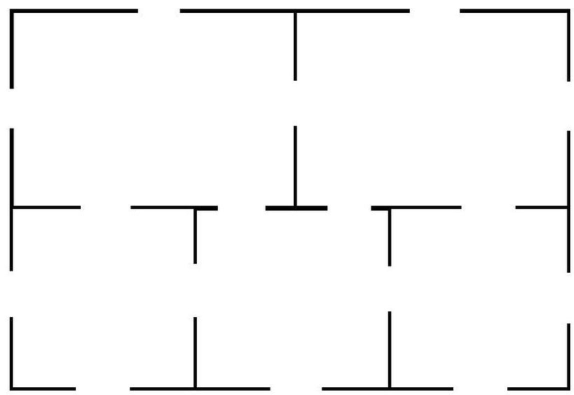
2. It is said that graph theory was born in Königsberg in 1736. Located on the Pregel river, the parts of the city were linked by seven bridges as shown below. The citizens wondered whether they could leave home, cross every bridge exactly once, and return home.



(a) If we transform the figure above into a graph, what does it look like?

(b) Can you find such a path for the citizens of Königsberg? If yes, draw it. If no, explain why not.

3. Consider the following five room apartment.

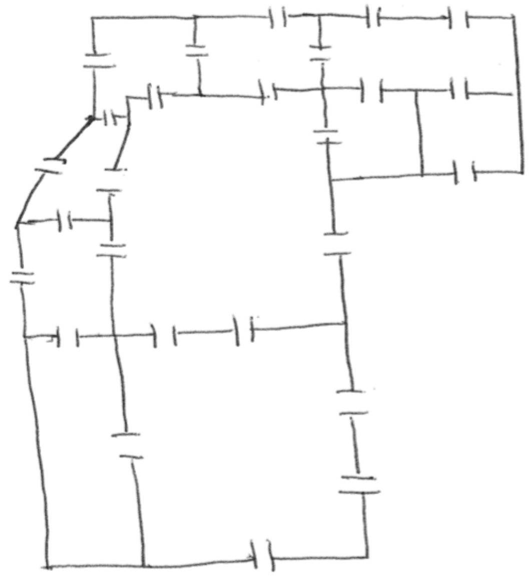


(a) If we make this floor plan into a graph, what does it look like?

(b) Can you find a continuous line that pass through each door exactly once? If yes, draw it. If no, explain why not.

(c) Now we are allowed to close doors of the apartment. After closing at least how many doors we can find a continuous line that passes through each door exactly once?

4. Consider the following apartment with doors.



(a) Without transforming the floor plan into a graph, can you tell whether there is a continuous line that pass through each door exactly once?

(b) At least how many doors are needed to be closed to have a continuous line that passes through each door exactly once?

5. Can you find anything in common for those Eulerian graphs, and for those graphs that are not Eulerian? From these examples, can you summarize the conditions needed for a graph to be Eulerian?

6. Is it possible to draw a floor plan with an odd number of exterior doors and an even number of doors in each room? If yes, draw the graph. If no, state the reason.

7. Is it possible to draw a floor plan with an odd number of exterior doors and the an even number of rooms with an odd number of doors? If yes, draw the graph. If no, state the reason.

8. Explain why the number of people in the world who have met an odd number of people is even.

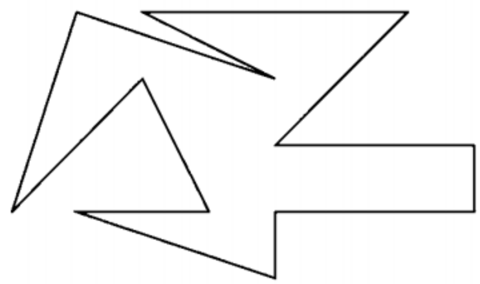
3 The Art Gallery Problem

A modern gallery has the shape of a simple polygon in the plane. Stationary guards are needed to watch the gallery.

1. Suppose the gallery has N corners and one stationary guard is enough to watch the gallery (regardless of the exact shape of the polygon). What is the largest possible value for N?

2. Suppose the gallery has N corners and we need at lest two stationary guards to watch the gallery (regardless of the exact shape of the polygon). What is the smallest possible value for N? Draw an example for that.

3. If the shape of the modern gallery is as below, what's the maximum number of stationary guards that may be needed to watch the gallery?



4. Draw a 9-corner gallery that needs at least 3 guards.

5. If the gallery has N corners, what's the maximum number of stationary guards that may be needed to watch the gallery? Explain you answer.