

TAXI CAB GEOMETRY

Washington University Math Circle

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GRID CITY Adam, Brenna, Carl, Dana, and Erik live in Grid City where each city block is exactly 300 feet wide. All five were in Middle School last year, but now they attend Grid City High School. The Middle School (MS) is located 6 blocks east and 4 blocks north of the High School (HS). Using the information below, plot **all possible** locations for everyone's homes on separate sheets.

A. Adam lives 5 blocks (1500 ft) from HS.

B. Last year Brenna walked 7 blocks (2100 ft) to MS, this year she only must walk 3 blocks (900 ft) to HS.

C. Carl uses his skateboard to get to school. He says that it is exactly the same distance from his apartment to HS as it was from his apartment to MS last year.

D. Dana rides her bike to school. She complains that this year she must ride 4 blocks (1200 ft) further every morning than when she biked to MS last year. The exact opposite is true for her best friend Denise, her bike ride to school is now 1200 ft shorter. Graph all possible locations for both **Dana and Denise**. **Bonus:** If Dana and Denise live 4 blocks apart, how many ordered pairs. [Dane, Denise] represent the locations for their two houses? What is they live 5 blocks apart?

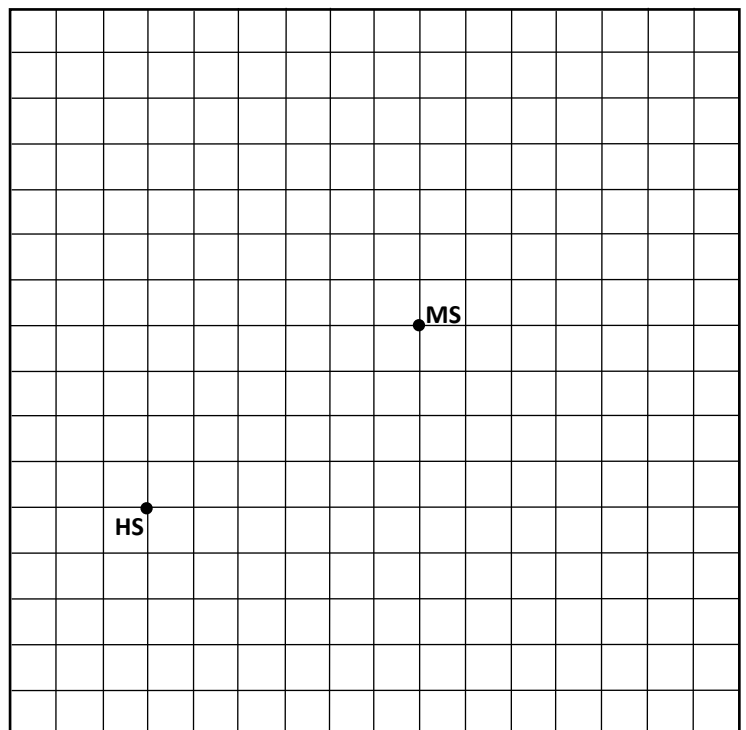
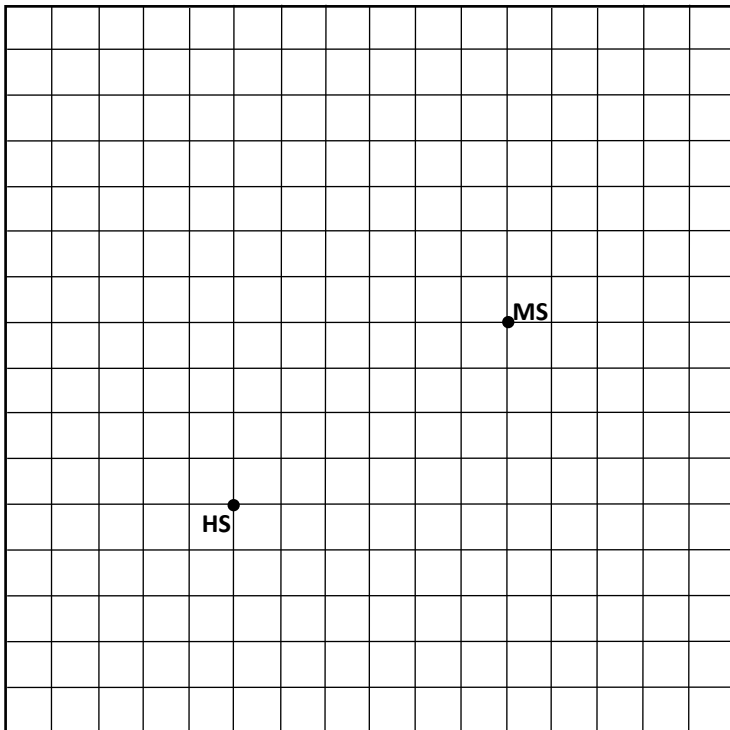
E. Last year Erik's best friend Earl moved to the suburbs, but finished the school year in the city. After school Earl walked from MS to Erik's house, and later he walked from Erik's house to the HS to catch a ride home with his sister after her basketball practice. Earl's total walking distance was 14 blocks (4200 ft). Graph all possible locations for **Erik's** house.

A. ADAM

Adam lives 5 blocks (1500 ft) from HS.

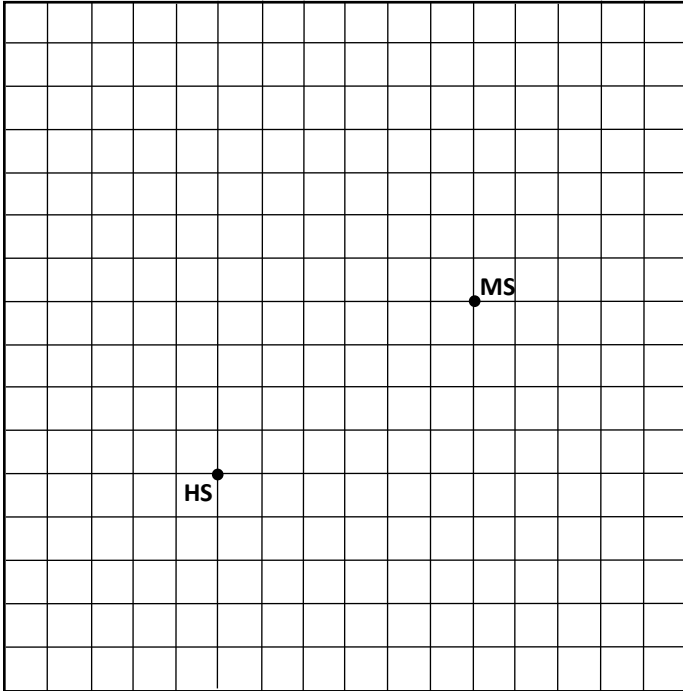
B. BREANNA

Last year Brenna walked 7 blocks (2100 ft) to MS, this year she only must walk 3 blocks (900 ft) to HS.



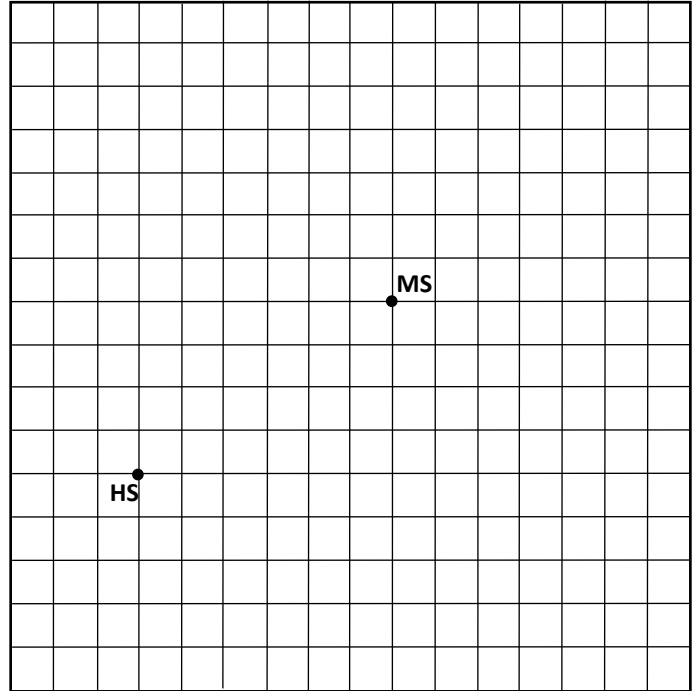
C. CARL

Carl uses his skateboard to get to school. He says that it is exactly the same distance from his apartment to HS as it was from his apartment to MS last year.



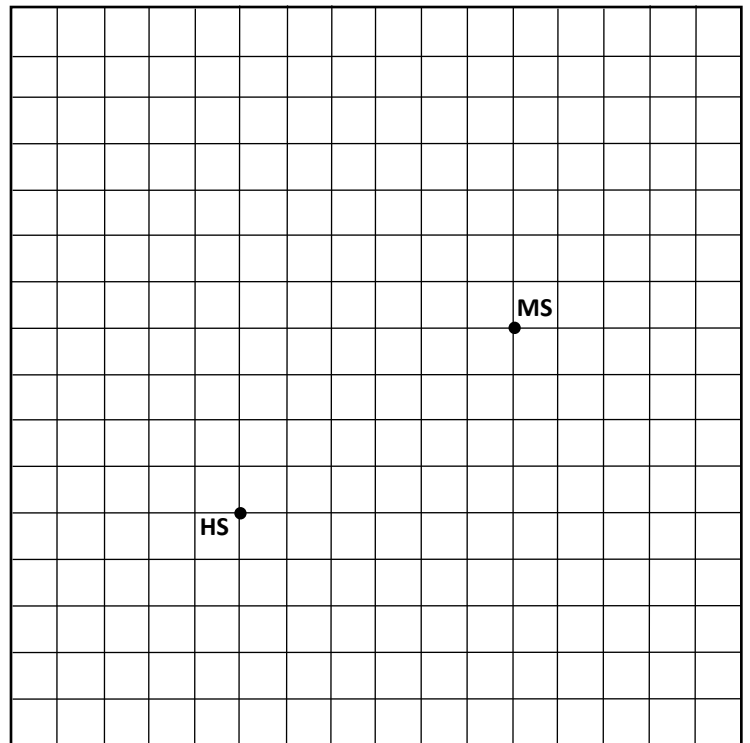
D. DANA & DENISE

Dana rides her bike to school. She complains that this year she must ride 4 blocks (1200 ft) further every morning than when she biked to MS last year. The exact opposite is true for her best friend Denise, her bike ride to HS is now 1200 ft shorter than her ride to MS had been. Graph all possible locations for both Dana and Denise.

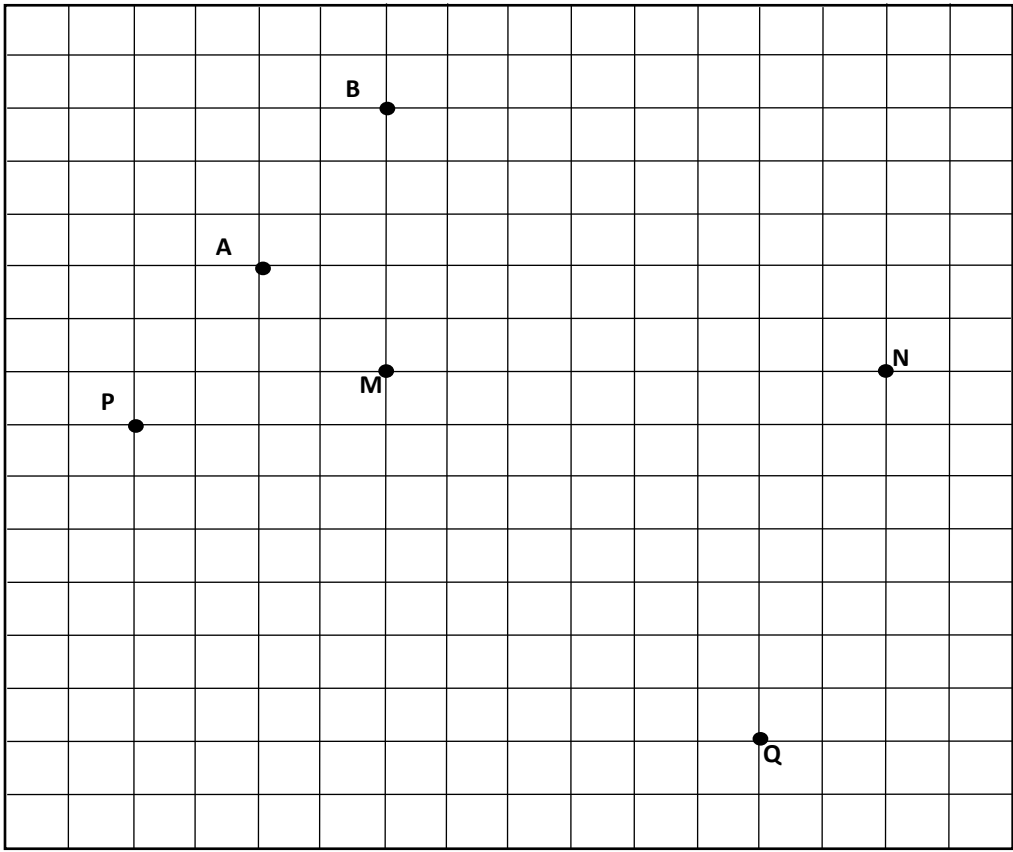


E. ERIC

Last year Erik's best friend Earl moved to the suburbs, but finished the school year in the city. After school Earl walked from MS to Erik's house, and later he walked from Erik's house to HS to catch a ride home with his sister after her basketball practice. Earl's total walking distance was 14 blocks (4200 ft). Graph all possible locations for Erik's house.



DISTANCES IN TAXI CAB GEOMETRY



What is the distance from A to B: in Euclidean Geometry? _____ in Taxicab Geometry? _____

What is the distance from P to Q: in Euclidean Geometry? _____ in Taxicab Geometry? _____

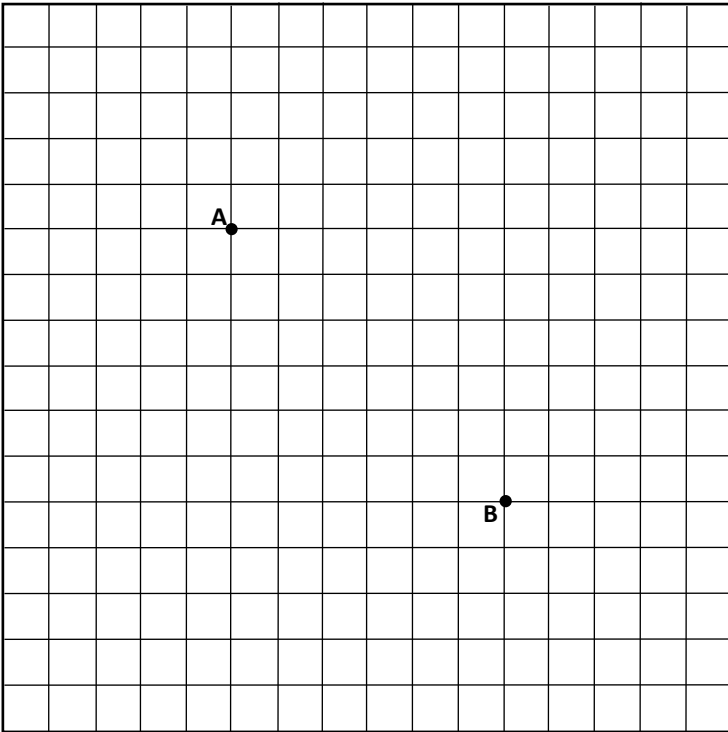
What is the distance from M to N: in Euclidean Geometry? _____ in Taxicab Geometry? _____

How many 'paths' of minimum distance are there from A to B: in EG? _____ in TCG? _____

[For Taxi Cab paths, "stay on the roads".]

BONUS: How many 'paths' of minimum distance are there from P to Q: in EG? _____ in TCG? _____

CIRCLES IN TAXI CAB GEOMETRY DEFINITION: With center C and radius R, a circle is the set of all points R units from C. For this worksheet, each center will be a lattice point.



Euclidean Geometry

Mark 12 points that are exactly 5 units from point A. Then draw the Euclidean circle with center A and radius 5.

Hint: Use a 'well-known' Pythagorean triplet.

Taxi Cab Geometry

Mark ALL lattice points that are exactly 5 units from point B. Then draw the Taxicab Geometry circle with center B and radius 5.

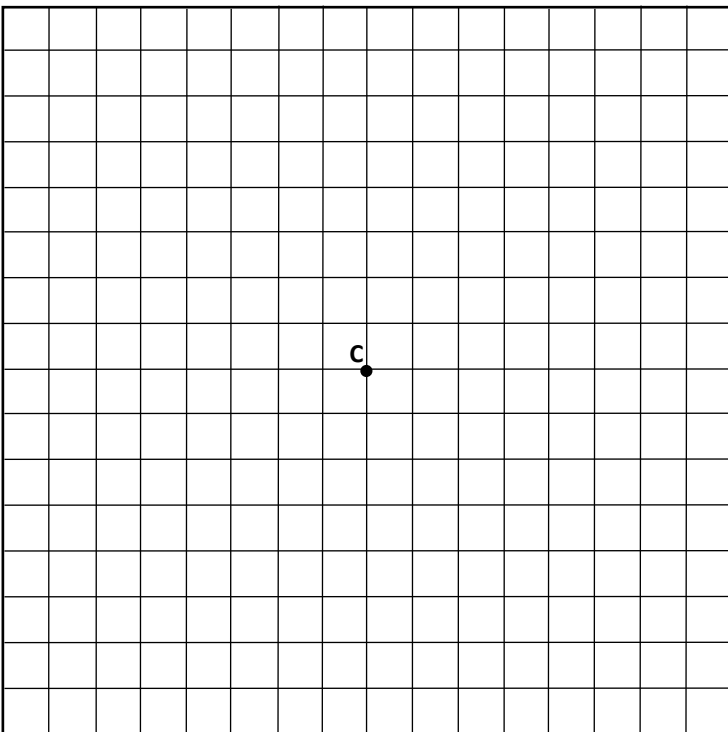
In TCG, how many lattice points are on this circle? _____

In TCG, how many lattice points are on any circle with radius R and a center on a lattice point? _____

In EG, what is the definition of Pi? *Pi* is _____ To 4 decimal places, what is the EG value of Pi? _____

For circle B, what is the TCG diameter D _____ and circumference C _____? Compute $C/D =$ _____.

Do you expect this ratio to be constant for all TCG circles? _____ Check C/D for some of the circles below.



In EG, two distinct circles can intersect in how many points? [list all answers] _____

In TCG, draw a circle with center C and radius 6.

Draw a TCG circle with center D and radius 2 which intersects circle C in exactly 2 lattice points.

Draw a TCG circle with center E and radius 1 which intersects circle C in exactly 1 lattice point.

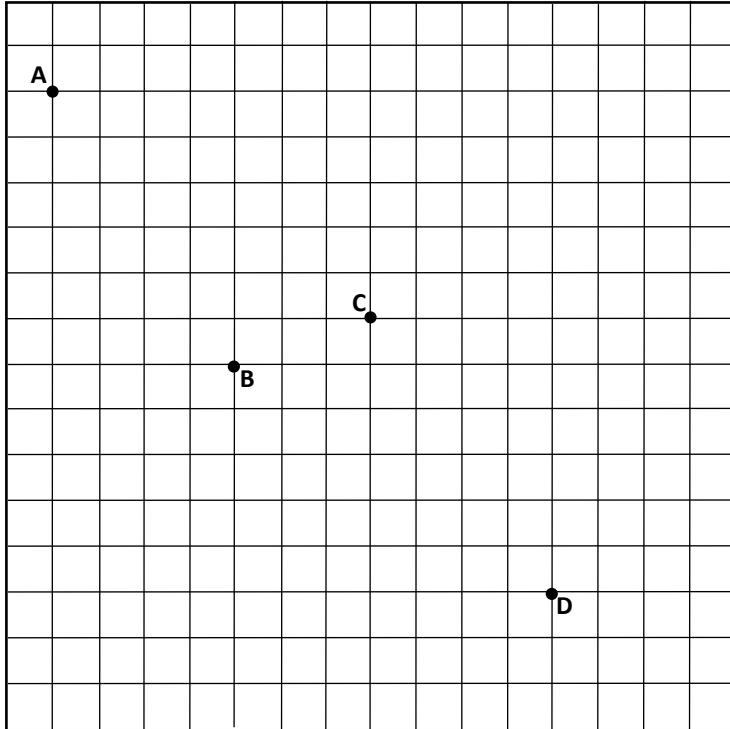
Draw a TCG circle with center F and radius 3 which intersects circle C in more than 4 lattice points.

If centers C and G are distinct, what is the maximum number of intersection points (lattice points) of circle C with radius 6 and any TCG circle with center G? _____

Draw a TCG circle with center H which intersects circle C in exactly 4 lattice points. Must all such circles have the same radius? _____

EQUI-DISTANT FROM TWO POINTS

Euclidean Geometry

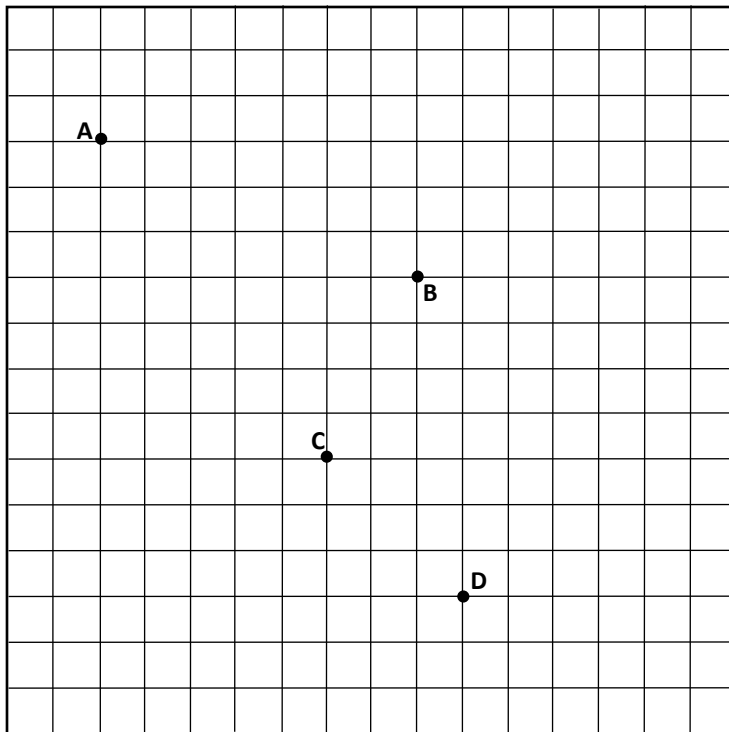


Mark at least three lattice points which are exactly the same distance from A and B. Then draw ALL points that are the same distance from A and B. What is the name of this set of points? _____

Taxicab Geometry

On the grid, draw all points that are the same TCG distance from C and D. [*Be careful!*]

[*In both EG and TCG, congruent circles can be useful in finding equidistant points.*]



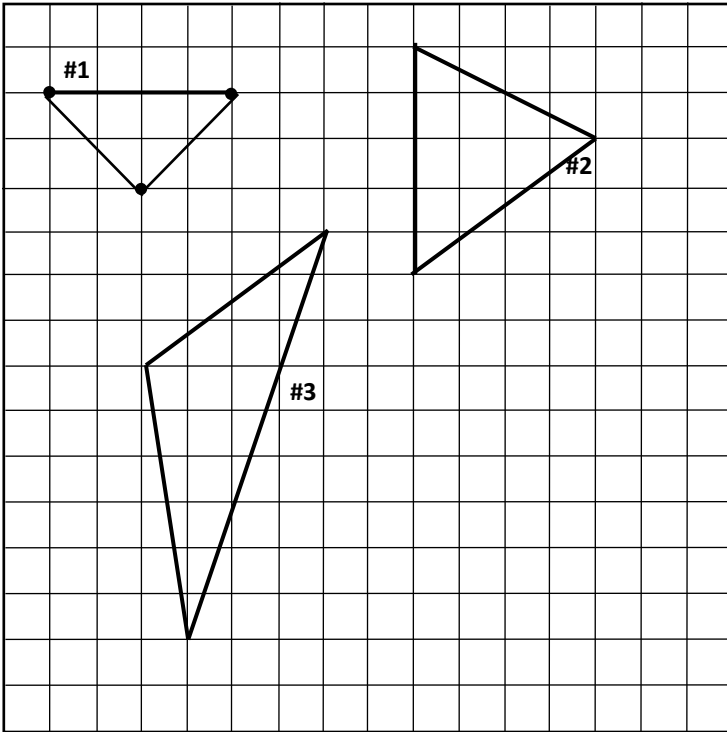
Taxicab Geometry

On the grid, draw all points that are the same TCG distance from A and B.

On the grid, draw all points that are the same TCG distance from C and D. [*Be careful!*]

TRIANGLES

We need a few TCG DEFINITIONS:



TCG Angle measure: If an angle measures x degrees in Euclidean, then it measures x degrees in TCG. *Note: There is a "better" way to measure angles in TCG. For simplicity, today we "borrow" the Euclidean notion. However, check the references to learn how a radian-like measure of Taxicab angles is useful in Taxicab geometry and trigonometry.*

TCG Equilateral triangle: A triangle with 3 sides of the same length. [Angles need not be congruent.]

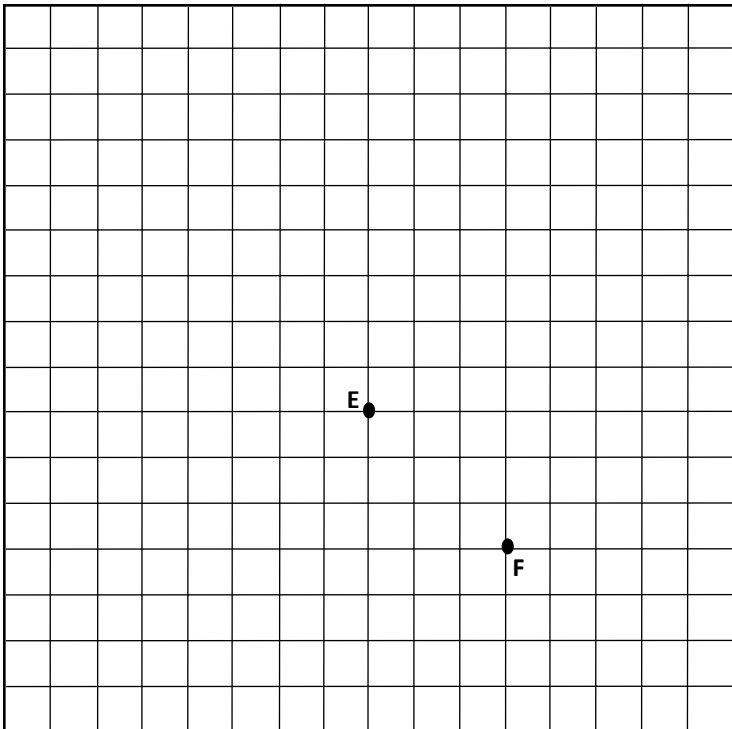
TCG Isosceles triangle: A triangle with exactly 2 sides of the same length. [It need not have congruent angles.]

Identify each triangle as Equilateral, Isosceles, or Scalene, in both EC and TCG.

#1: EC: _____; TCG: _____

#2: EC: _____; TCG: _____

#3: EC: _____; TCG: _____



Challenges:

1. Determine all eight points G such that EFG is an equilateral TCG triangle. What types of Euclidean Geometry triangles are represented among these EFG triangles?

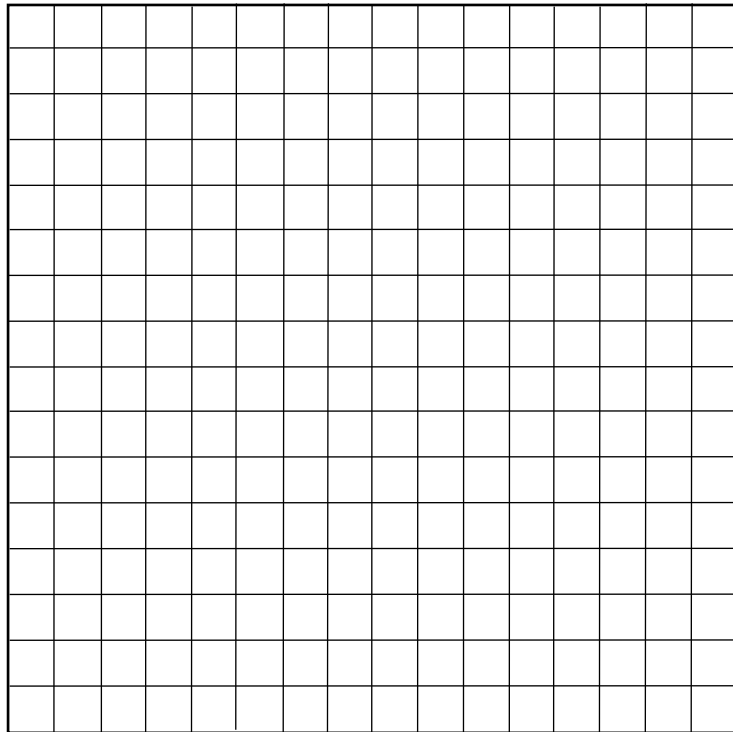
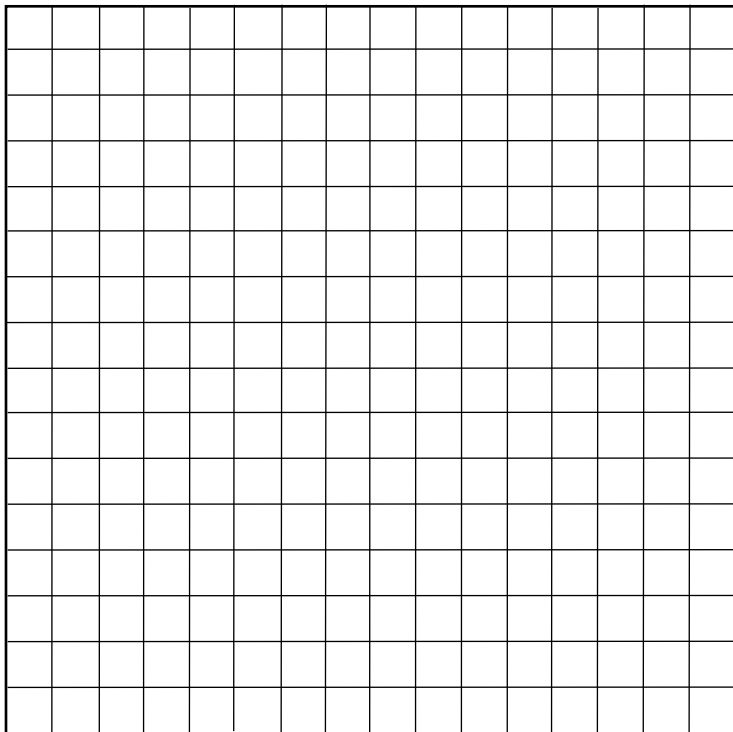
2. Draw a TCG equilateral triangle ABC which is an EC scalene triangle. [Consider using TCG circles or perpendicular bisectors.]

3. Draw a triangle PQR which is scalene in both TCG and EG such that PQ is the longest side in EG while PR is the longest side in TCG.

HYPERBOLAS

Refer to the Dana and Denise problem from Part D on Page 2 (above). Be sure to solve that problem before attempting this page.

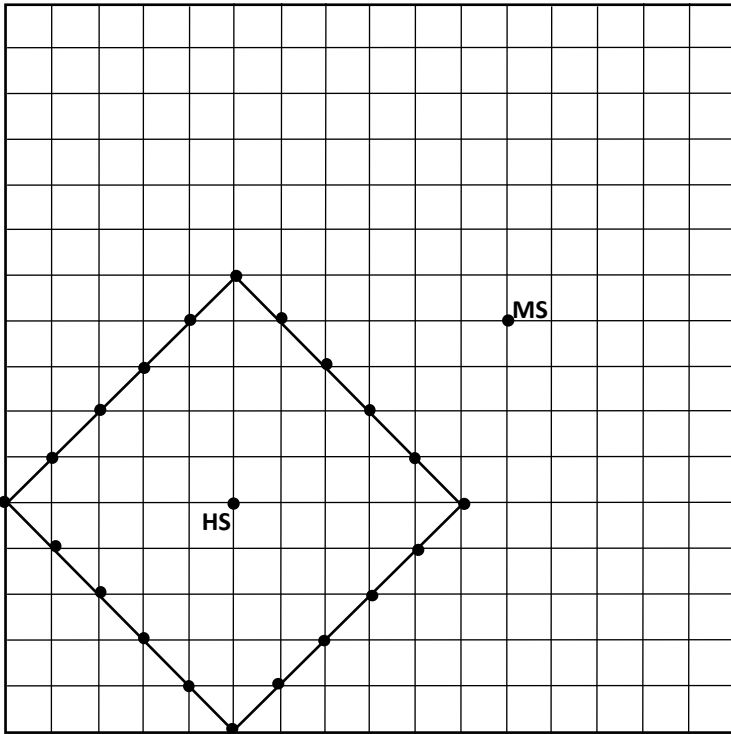
With the Middle School and High School 10 blocks apart and with Dana 4 blocks closer to MS than HS [and Denise 4 blocks closer to HS than MS], you found the two branches of a TCG Hyperbola. By changing only the relative locations of MS and HS [keeping the 10 and 4 the same], three more **distinctly shaped** hyperbolas [each with two branches] are possible. Can you find the other three?



SOLUTIONS

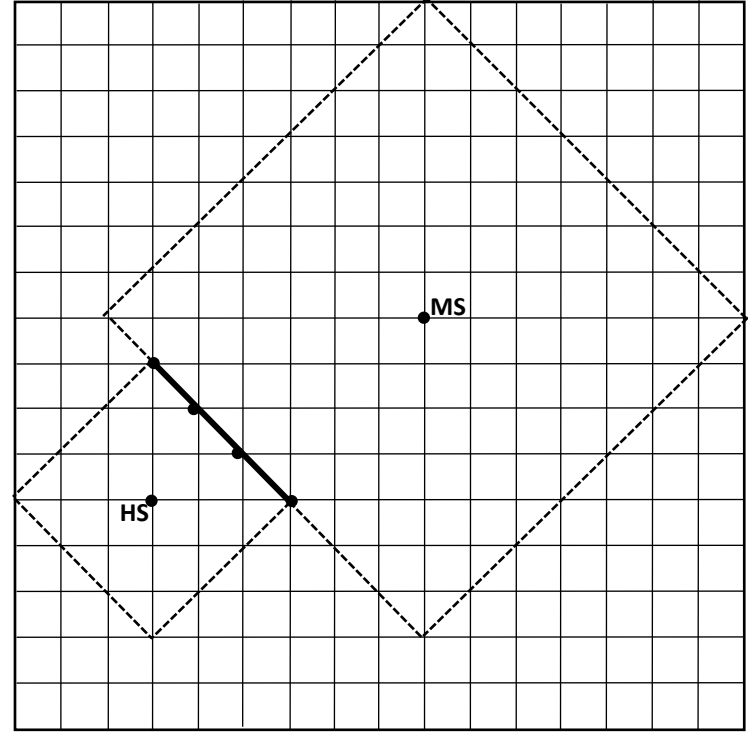
A. ADAM

A Taxi Cab *circle*.



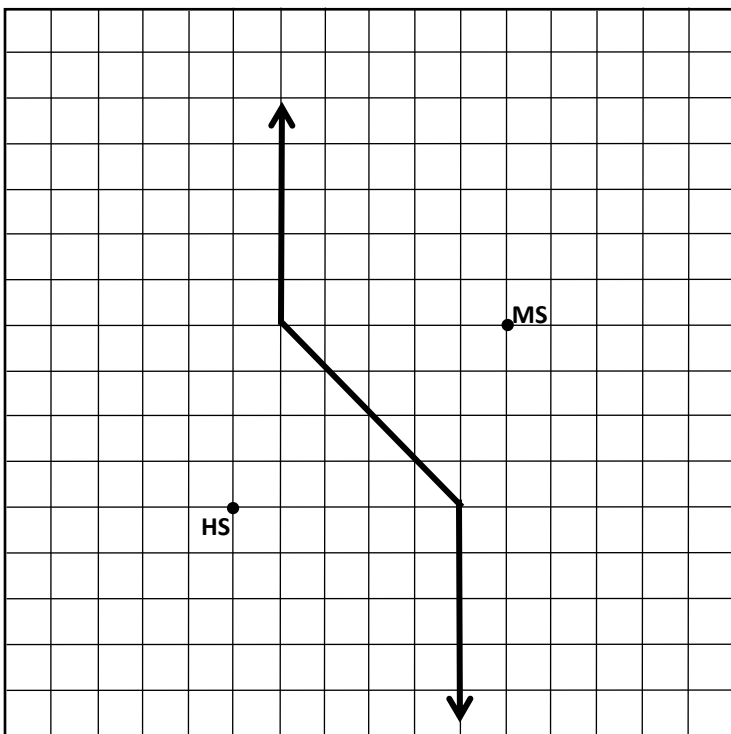
B. BREANNA

The *intersection of two circles*.



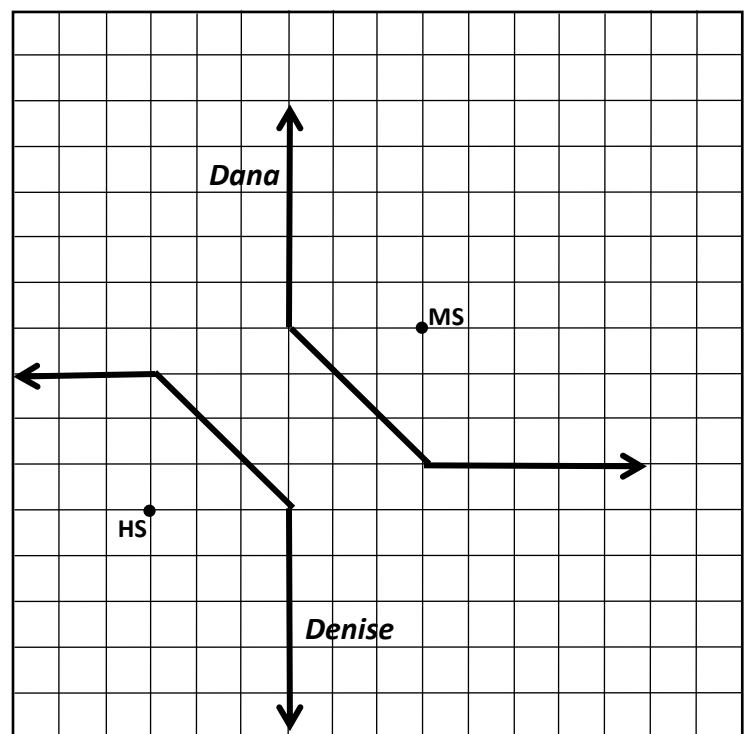
C. CARL

A *perpendicular bisector*.



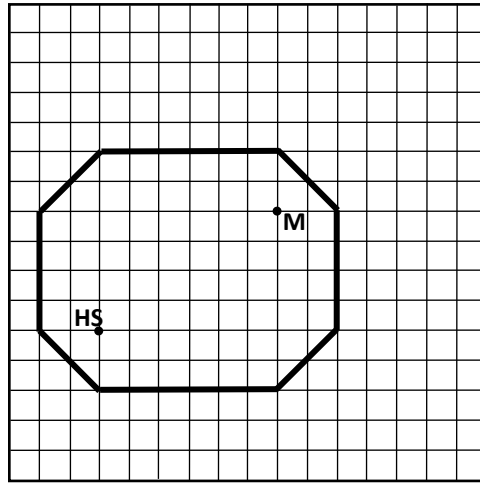
D. DANA & DENISE

Two branches of a *hyperbola*.



Dana and Denise bonus questions: 4 blocks apart? $2+2+3+4 = \mathbf{11 \text{ ordered pairs}}$; 5 blocks apart? $2+1+2+2 = \mathbf{7 \text{ ordered pairs}}$;

E. ERIC An *ellipse*.



DISTANCES

What is the distance from A to B: in Euclidean Geometry? $\sqrt{13} \approx 3.6$; in Taxicab Geometry? 5

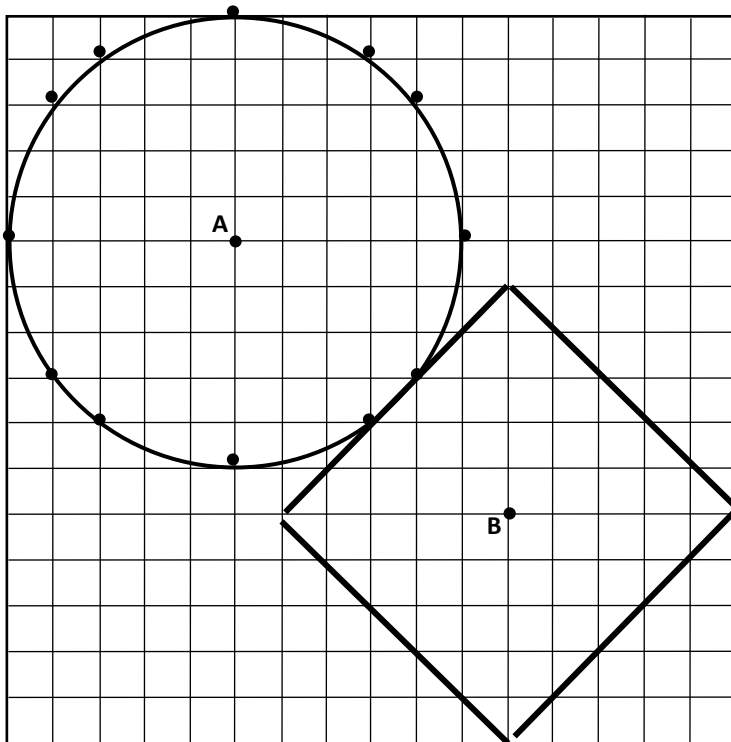
What is the distance from P to Q: in Euclidean Geometry? $\sqrt{136} \approx 11.7$; in Taxicab Geometry? 16

What is the distance from M to N: in Euclidean Geometry? 8 ; in Taxicab Geometry? 8

How many 'paths' of minimum distance are there from A to B: in EG? One ; in TCG? 10

BONUS: How many 'paths' of minimum distance are there from P to Q: in EG? One ; in TCG? $\binom{16}{6} = 8008$

CIRCLES IN TAXI CAB GEOMETRY



Euclidean Geometry

Mark 12 points that are exactly 5 units from point A. Then draw the Euclidean circle with center A and radius 5.

Taxi Cab Geometry

Mark ALL lattice points that are exactly 5 units from point B. Then draw the Taxicab Geometry circle with center B and radius 5.

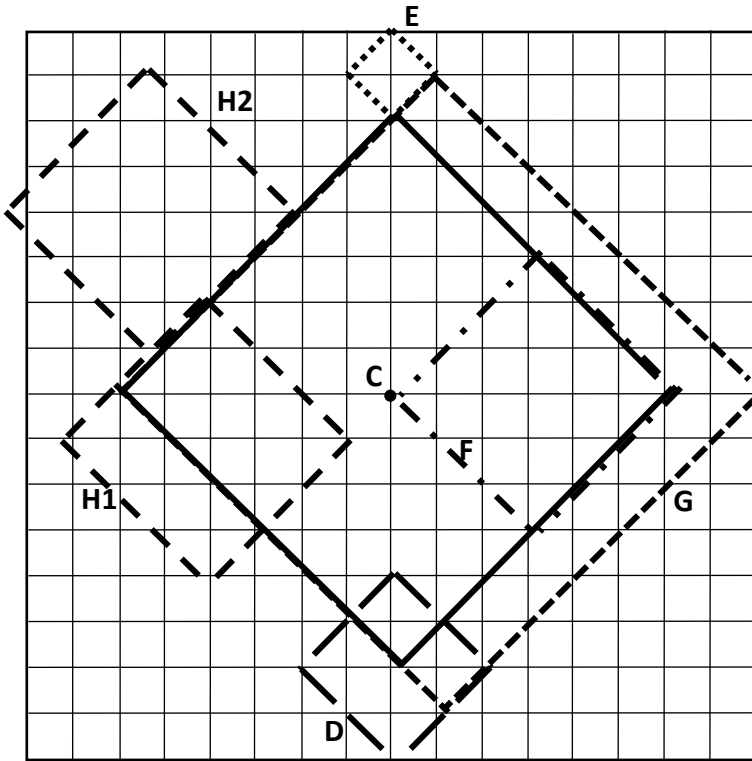
In TCG, how many lattice points are on this circle? 20

In TCG, how many lattice points are on any circle with radius R and a center on a lattice point? 4R

In EG, what is the definition of Pi? *Pi is the ratio of circumference to diameter.* To 4 decimal places, what is the EG value of Pi? 3.1416

For circle B, what is the TCG diameter D and circumference C? Compute $C/D = 40/10 = 4$

Do you expect this ratio to be constant for all TCG circles? YES it is



In EG, two distinct circles can intersect in how many points? [list all answers] 0 or 1 or 2

In TCG, draw a circle with center C and radius 6.

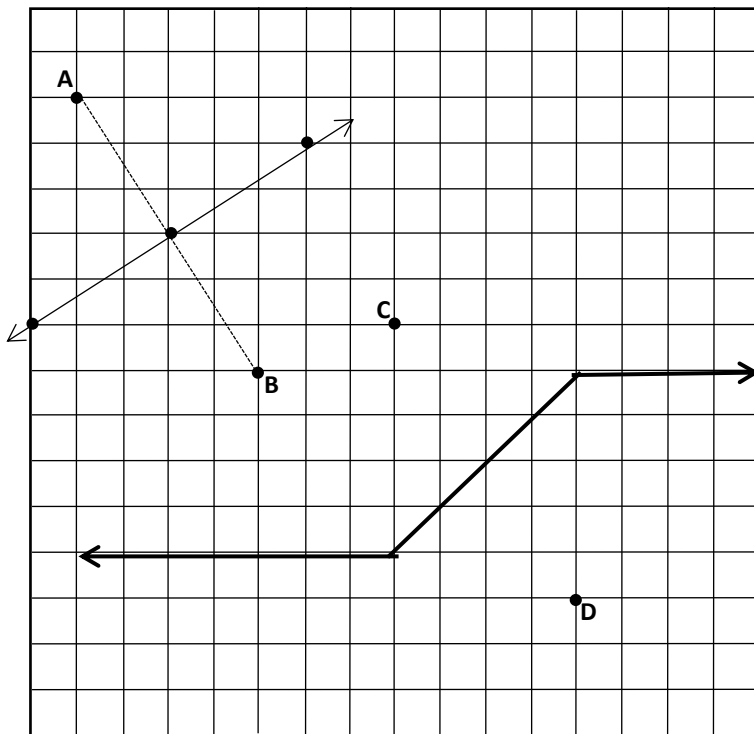
Draw a TCG circle with center D and radius 2 which intersects circle C in exactly 2 lattice points.

Draw a TCG circle with center E and radius 1 which intersects circle C in exactly 1 lattice point.

Draw a TCG circle with center F and radius 3 which intersects circle C in more than 4 lattice points.

If centers C and G are distinct, what is the maximum number of intersection points (lattice points) of circle C with radius 6 and any TCG circle with center G? 13

Draw a TCG circle with center H which intersects circle C in exactly 4 lattice points. Must all such circles have the same radius? NO



EQUI-DISTANT FROM TWO POINTS

Euclidean Geometry

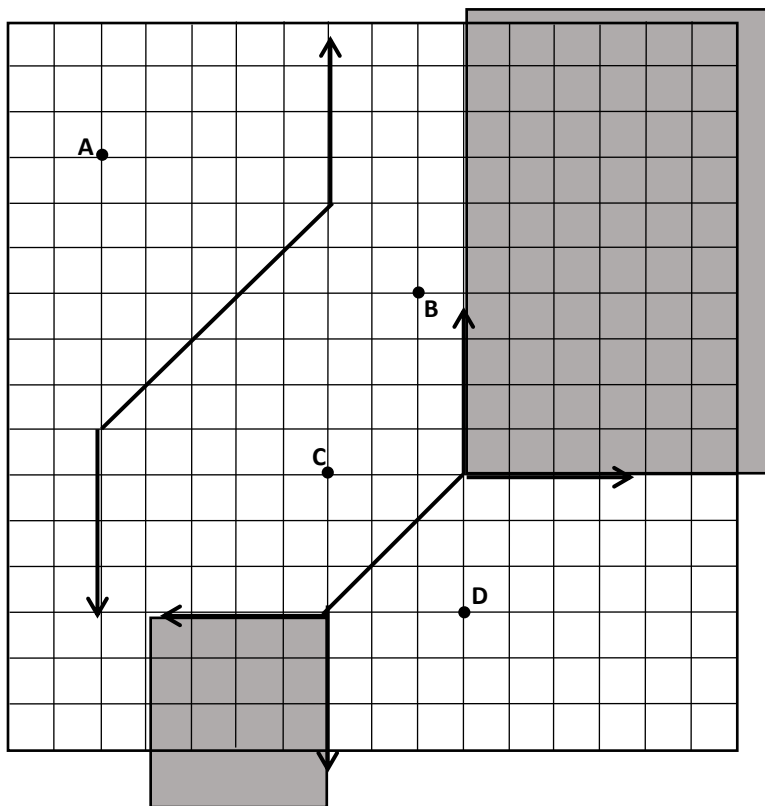
Mark at least three lattice points which are exactly the same distance from A and B. Then draw ALL points that are the same distance from A and B. What is the name of this feature?

Perpendicular Bisector

Taxicab Geometry

On the grid, draw all points that are the same TCG distance from C and D. [Be careful!]

[In both EG and TCG, congruent circles can be useful in finding equi-distant points.]



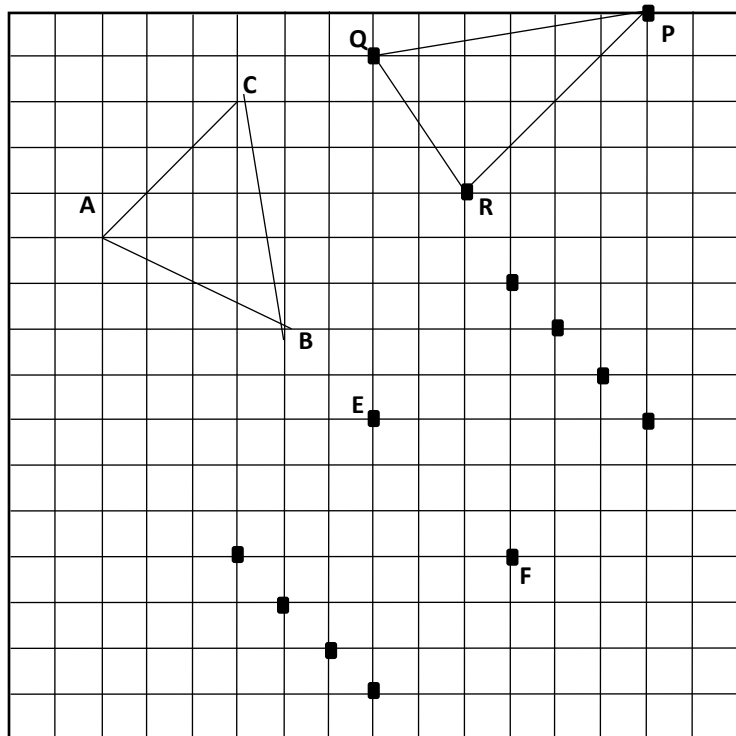
Taxicab Geometry

On the grid, draw all points that are the same TCG distance from A and B.

On the grid, draw all points that are the same TCG distance from C and D. *[Be careful!]*

TRIANGLES

#1: EC: Isosceles; TCG: Equilateral! **#2:** EC: Isosceles; TCG: Scalene **#3:** EC: Scalene ; TCG: Isosceles



Challenges:

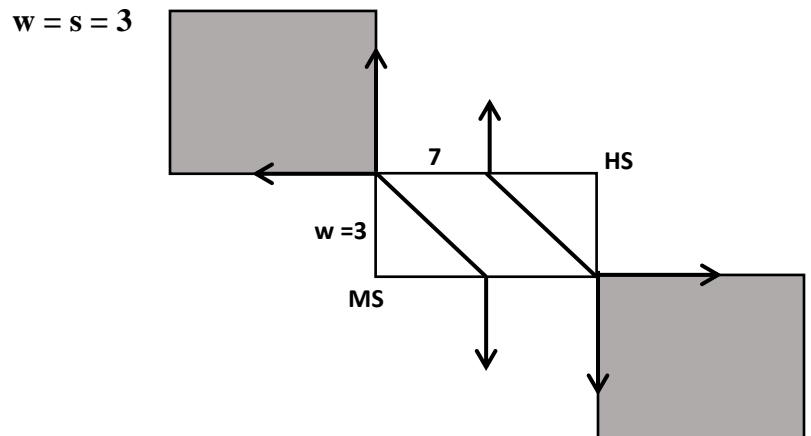
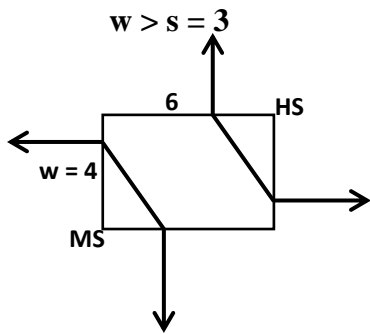
1. Determine all eight points G such that EFG is an equilateral TCG triangle. What types of EG triangles are represented among all EFG triangles? Scalene and right isosceles
2. Draw a TCG equilateral triangle ABC which is an EC scalene triangle. [Consider using TCG circles or perpendicular bisectors.] Many solutions are possible.
3. Draw a triangle PQR which is scalene in both TCG and EG such that PQ is the longest side in EG while PR is the longest side in TCG.

HYPERBOLAS

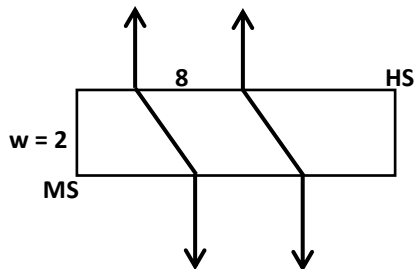
Let D equal the distance from MS to HS. Let k equal how much closer Dana is to MS than to HS. Define: $s = (D - k)/2$.

In this problem, $CD = 10$, $k = 4$, and $s = 3$. The key is to compare $s = 3$ to the shorter side (w) of the rectangle formed by MS and HS. Diagrams are not drawn accurately to scale.

Four cases:



$0 < w < s = 3$



$w = 0$

