Symmetry Groups

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1 Shuffles

Problem 1

Let S_n be the set of all possible shuffles of n cards.

1. How many ways can we shuffle 6 cards?

2. How many ways can we shuffle n cards?

Problem 2

Given a shuffle of n cards, the order of the shuffle is the smallest number of times one needs to perform to reach the original order of cards.

We are given 6 cards indexed by $1, 2, 3, 4, 5, 6$.

1. Find a shuffle of order 2.

2. Find a shuffle of order 3.

3. Find a shuffle of order 4.

4. Find a shuffle of order 6.

5. Describe the composition of the order 2 shuffle and order 3 shuffle you have found.

6. Describe the composition of the order 4 shuffle and order 6 shuffle you have found.

Problem 3

Prove the following.

1. The composition of any two even order shuffles is again an even order shuffle.

2. The composition of any two odd order shuffles is an even order shuffles.

2 Plane Isometry

Problem 4

Consider an equilateral triangle in the plane.

1. Find a rotation on the plane that fixes the triangle.

2. Find all rotations on the plane that fix the triangle.

3. Prove that the composition any two above rotations is still a rotation.

4. Find a flip with respect to a line in the plan that fixes the triangle.

5. Prove that the composition of any two above flips is a rotation.

6. Prove that the composition of the above rotation and flip is a flip.

Problem 5

1. Find all the triangles that have nontrivial rotational symmetries.

2. Find all the triangles that have nontrivial flip symmetries.

3. Find all the triangles that have two nontrivial flip symmetries.

Problem 6

Consider a square in the plane.

1. Find all the rotations on the plane that fix the square.

2. Prove that the composition any two above rotations is still a rotation.

3. Final all the rotations on the plane that fix the square.

4. Prove that the composition of any two above flips is a rotation.

5. Prove that the composition of the above rotation and flip is a flip.

Problem 7

1. Find all the rotations that fix the rectangle of size $3in \times 4in$.

2. Find all the flips that fix the rectangle of size $3in \times 4in$.

3. Find all the parallelograms that have nontrivial rotational symmetries.

4. Find all the parallelograms that have nontrivial flip symmetries.

5. Find all the trapezoids that have nontrivial rotational symmetries.

6. Find all the trapezoids that have nontrivial flip symmetries.

Problem 8

1. Does there exist a trapezoid that have exactly 2 nontrivial rotational symmetries?

2. Does there exist a trapezoid that have exactly 3 nontrivial flip symmetries?

3. Does there exist a trapezoid that have exactly 1 nontrivial rotational symmetry and 4 nontrivial flips symmetries?

4. Does there exist a figure in the plane that have exactly 1 nontrivial rotational symmetry and 3 nontrivial flip symmetries?

Problem 9

1. Find a figure in the plane that have exactly 4,5,6 nontrivial rotational symmetries.

2. Find a figure in the plane that have exactly n nontrivial rotational symmetries.

3. Find a figure in the plane that have exactly 5,6,7 flip symmetries.

4. Find a figure in the plane that have exactly n flip symmetries.

Problem 10

1. Prove that if a figure in the plane has no translational symmetries, the figure cannot have rotational symmetries about two distinct points.

2. Prove that if a figure in the plane has no translational symmetries, the figure cannot have a rotational symmetry about a point A and a flip symmetry with respect to any line that not through A.

3. Prove that if a figure in the plane has no translational symmetries, it can only contain flip symmetries with respect to lines through one point.

4. Prove that if a figure in the plane has no translational symmetries, and has exactly n number of flip symmetries, it has exactly $n-1$ number of nontrivial rotational symmetries.

Problem 11

Prove the following theorem of Leondardo da Vinci (1452-1519).

If a figure in the plan has no translational symmetries, there is a natural number n such that one of the following cases happens.

1. The figure has exactly n number of flip symmetries and $n-1$ number of nontrivial rotational symmetries.
2. The figure has exactly $n-1$ number of nontrivial rotational symmetries and no flip symmetries.