AM-GM Inequality

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1 Warmup

Solving an inequality means to find all solutions to the inequality.

1. The length of the interval of solutions of the inequality $a \leq 2x + 3 \leq b$ is 10. What is $b - a$? (From AMC10 2010)

2. Solve $x^2 - 3x + 2 > 0$.

3. For which x do we have $3x < x^3$?

4. Can you find two positive integers x and y such that $x + y < 2\sqrt{xy}$?

2 Arithmetic mean and geometric mean

1. Consider a series of real numbers $a_1, \ldots, a_n$. If all the quantities had the same value, what would that value have to be in order to achieve the same total?

   The answer is the *arithmetic mean* (AM),

   $$AM = \frac{a_1 + \cdots + a_n}{n}.$$

2. Question: Suppose you have an investment which earns 10% the first year, 50% the second year, and 30% the third year. What is the average annual rate of return?

   If your answer is 30%, unfortunately, it is incorrect. The correct answer is the geometric mean of the three numbers.

3. Again consider a series of real numbers $a_1, \ldots, a_n$. If all the quantities had the same value, what would that value have to be in order to achieve the same product?

   The answer is then the *geometric mean* (GM),

   $$GM = \sqrt[n]{a_1 \cdots a_n}.$$

   Note that, in general, you can only find the geometric mean of positive numbers.

4. Question: What is the geometric mean rate of return for an investment that shows a growth in year 1 of 10% and a decrease the next year of 15%?

5. One alternative way to think about the GM is as follows. Given a set of positive real numbers $a_1, \dots, a_n$, if we look at these numbers on their log-scale, their AM is

$$\frac{\ln a_1 + \cdots + \ln a_n}{n}.$$

After transforming it back to the original scale, we have

$$\exp\left\{\frac{\ln a_1 + \cdots + \ln a_n}{n}\right\} = GM.$$

3 AM-GM Inequality

Now, what is the relationship between AM and GM? Recall the last warmup question. The answer is actually NO.

In general, for any two positive real numbers x and y, it is always true that $x + y \geq 2\sqrt{xy}$. This result shows that

$$AM = \frac{x+y}{2} \geq \sqrt{xy} = GM.$$

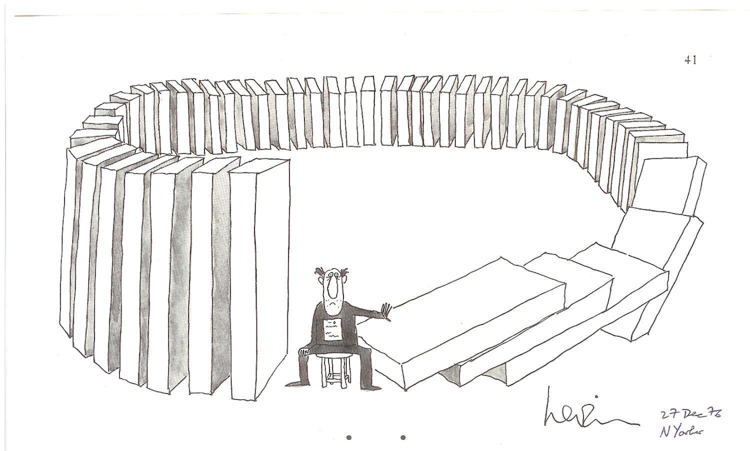
Is it true in general?

Let's draw a picture...

Now, let's consider the case beyond just two positive real numbers. Given n positive real numbers $a_1, \ldots, a_n$, do we still have $AM \geq GM$?

In the following, we will prove the result by mathematical induction. In case you are not familiar with the idea of induction, here it is. It has only 2 steps:

- Step 1. Show it is true for the **first** one.
- Step 2. Show that if **any one** is true then the **next one** is true.
- Then **all** are true.



Let us exercise this idea by proving

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

- Step 1. Show the statement is true for $n = 1$.

- Step 2. Show that if the statement is true for $n = k$ then it is also true for $n = k + 1$.

Now it is time to come back to the proof of $AM \geq GM$. By now, we know it is true for $n = 1$ and $n = 2$.

Without loss of generality, let us rescale a_i's so that $a_1 \cdots a_n = 1$. Then we just need to show

$$\frac{a_1 + \cdots + a_n}{n} \geq 1.$$

If all $a_i = 1$, the proof is trivial. Thus, let us assume at least one $a_i > 1$ and one $a_i < 1$, and we name these two values a_1 and a_2, i.e. $a_1 > 1$ and $a_2 < 1$. Now the induction is ready to begin ...

Assume that the AM-GM inequality is true for any $n-1$ positive numbers. Hence, by applying this to the set of $n-1$ numbers, $a_1 a_2$, $a_3, \ldots, a_n$, we have

$$\frac{a_1 a_2 + a_3 + \cdots + a_n}{n-1} \geq \sqrt[n-1]{(a_1 a_2) a_3 \cdots a_n} = 1.$$

That is,

$$a_1 a_2 + a_3 + \cdots + a_n \geq n - 1.$$

Recall our goal is to show for n numbers $a_1, \ldots, a_n$,

$$a_1 + a_2 + \cdots + a_n \geq n.$$

This would follow if $a_1 + a_2 - (a_1 a_2 + 1) \geq 0$. Can you complete the proof?

4 Application of the AM-GM inequality

1. Consider rectangles with fixed perimeter P. What is the one that has the largest area?

2. Consider rectangles with fixed area A. What is the one that has the minimum perimeter?

3. A farmer wants to fence in 60,000 square feet of land in a rectangular plot along a straight highway. The fence he plans to use along the highway costs \$2 per foot, while the fence for the other three sides costs \$1 per foot. How much of each type of fence will he have to buy in order to keep expenses to a minimum? What is the minimum expense?

4. Find the maximum of $f(x) = (1 - x)(1 + x)(1 + x)$ when $x \in (0, 1)$.

5. Find the smallest value of $x^2 + 4xy + 4y^2 + 2z^2$ for positive real numbers x, y, z with product 32.

6. For positive real numbers a, b, and c, prove

$$(a+b)(b+c)(c+a) \geq 8abc.$$

7. Prove *Padoa's Inequality*: If a, b, and c are the sides of a triangle, then

$$abc \geq (a+b-c)(b+c-a)(c+a-b).$$