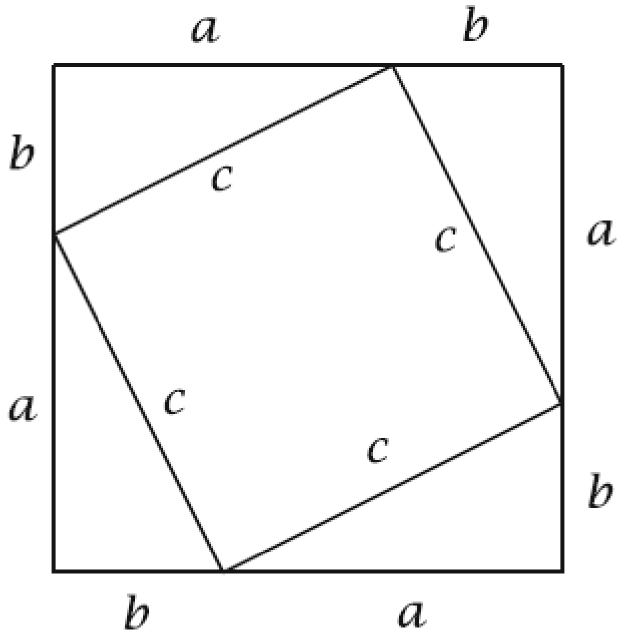
WUSTL Math Circle

Proofs Without Words

(based on Nelson, R. G., *Proofs Without Words*, MAA, 1993.)

Consider the following figure:



The big square has side length $a+b$, so its area is $(a+b)^2 = a^2+b^2+2ab$. This number must be the same as the sum of areas of pieces: a small square of area c^2 and four right triangles each of area $\frac{1}{2}ab$. Therefore

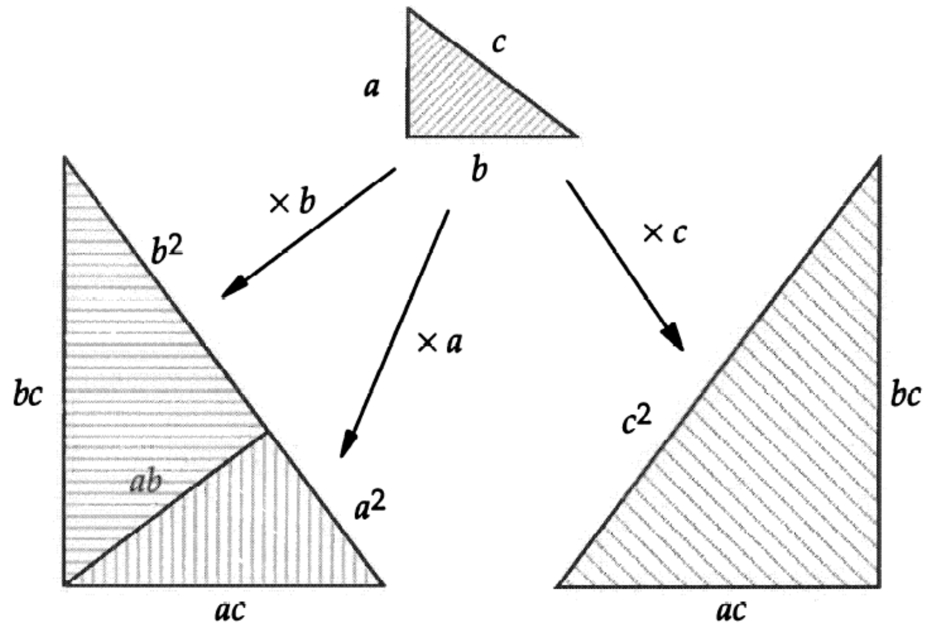
$$a^2 + b^2 + 2ab = c^2 + 4 \times \frac{1}{2}ab = c^2 + 2ab.$$

Canceling $2ab$ from both sides, we have

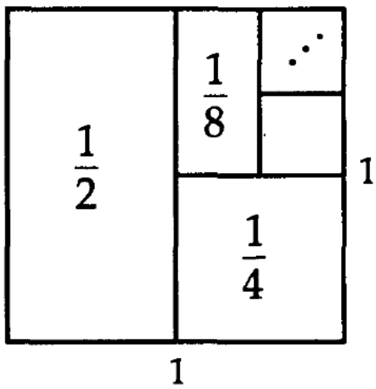
$$a^2 + b^2 = c^2.$$

This proves Pythagoras' Theorem: *the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides*.

1. What can you deduce from the following figure?



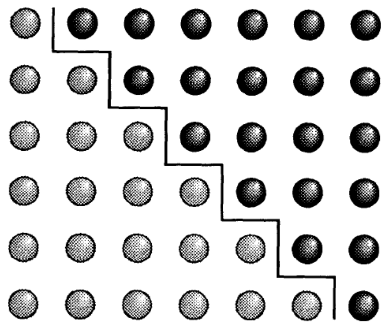
2. Use



to compute infinite series

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \ ?$$

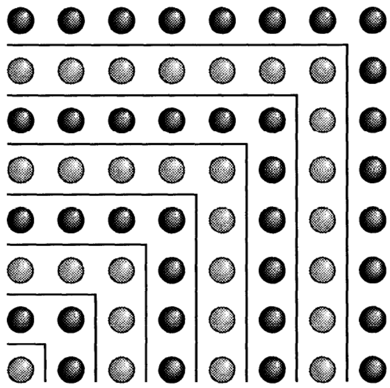
In



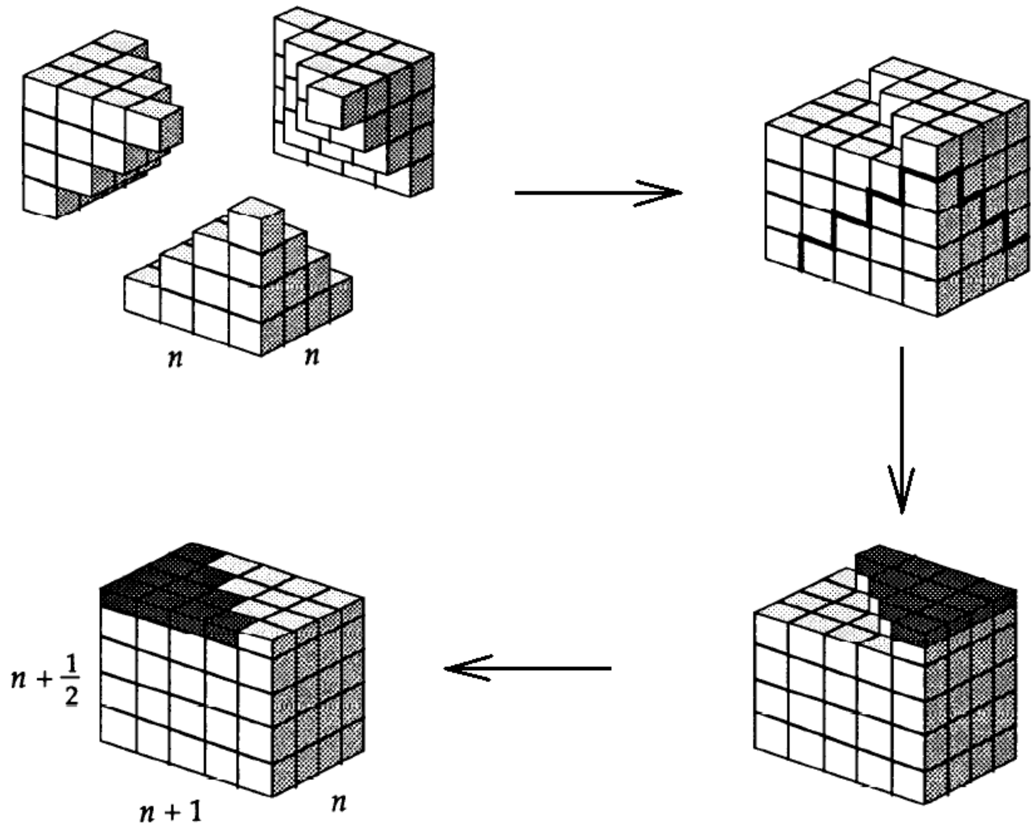
counting the number of balls in two ways proves that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

3. What can you deduce from the following figure?



4. Use

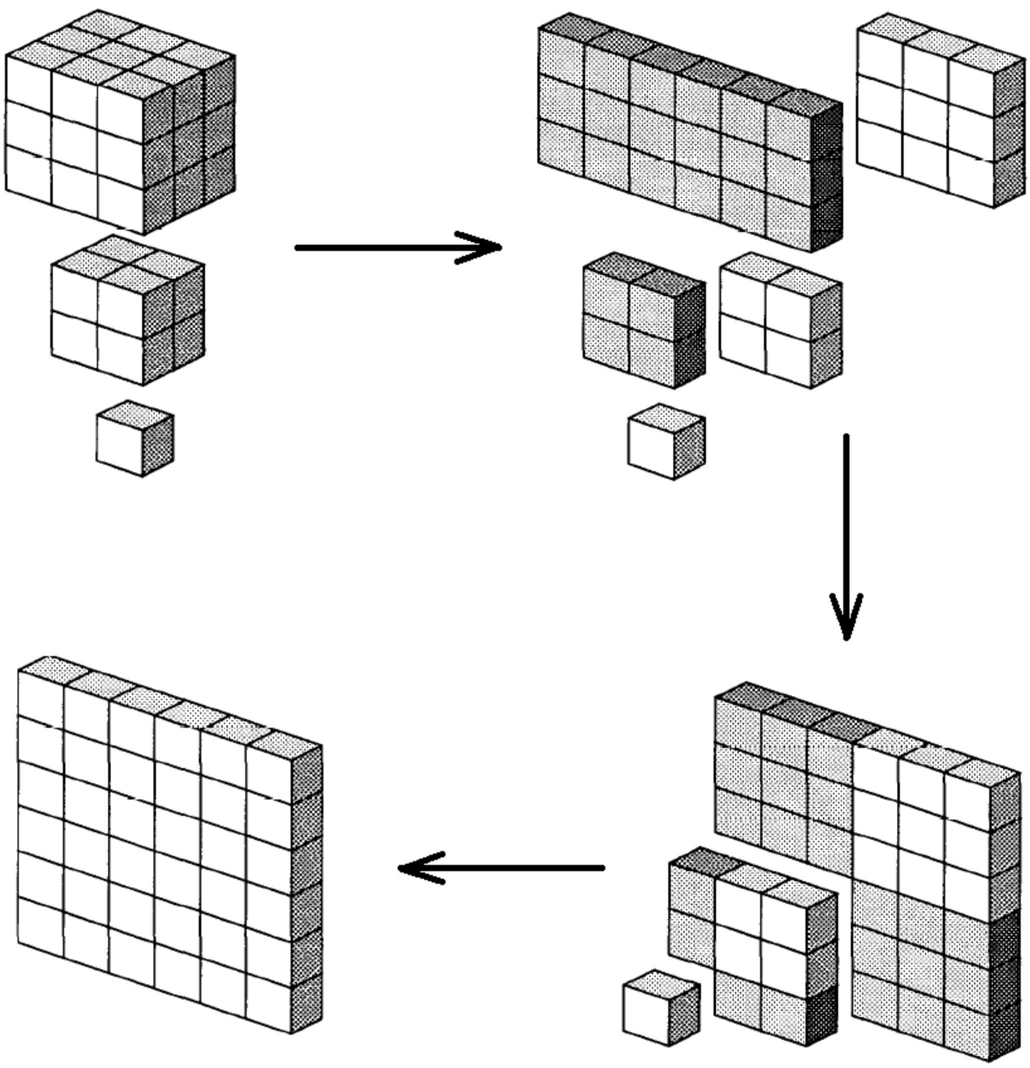


to find a formula for

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \ ?$$

(Answer is given in the last page.)

5. Use

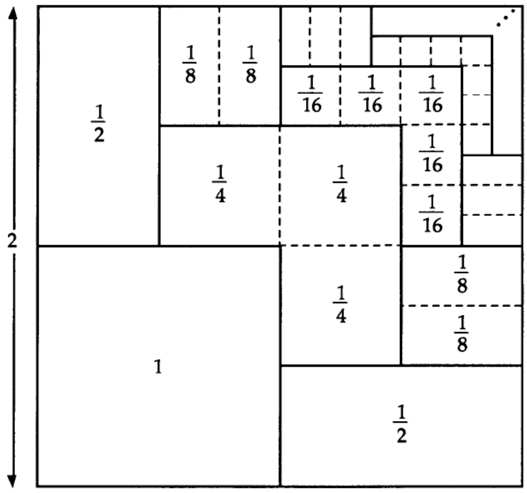


to find a formula for

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \ ?$$

(Answer is given in the last page.)

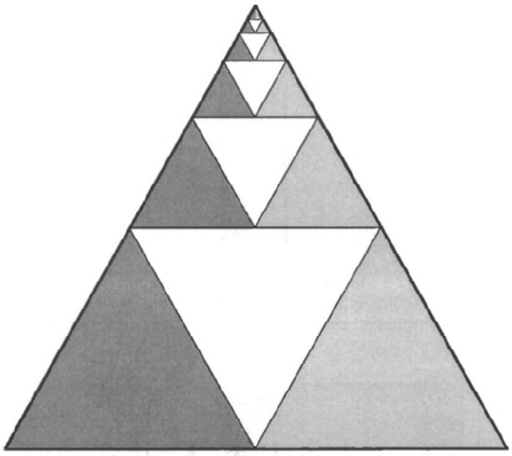
6. Use



to compute infinite series

$$1 + 2\frac{1}{2} + 3\frac{1}{2^2} + 4\frac{1}{2^3} + 5\frac{1}{2^4} + \cdots = \ ?$$

7. In

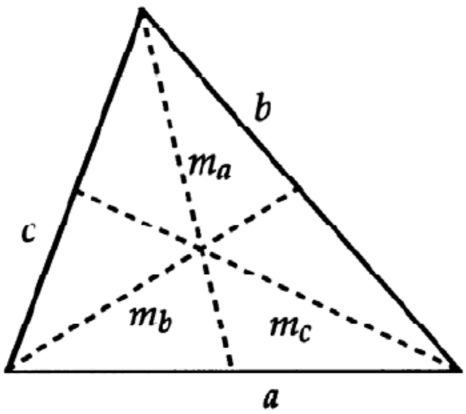


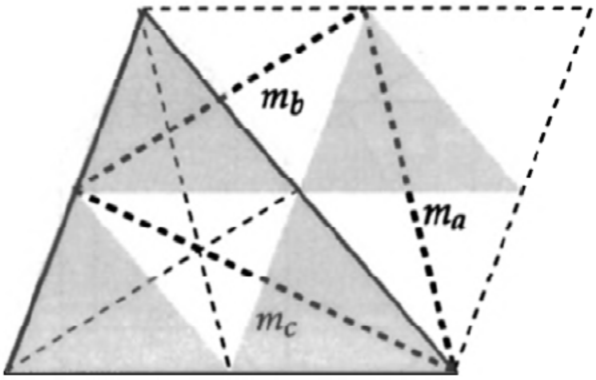
assume the big traingle has unit area. Compute the infinite series

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \ ?$$

In a triangle, *median* is a line segment joining a vertex to the midpoint of the opposite side. Each triangle has three medians.

8. Use

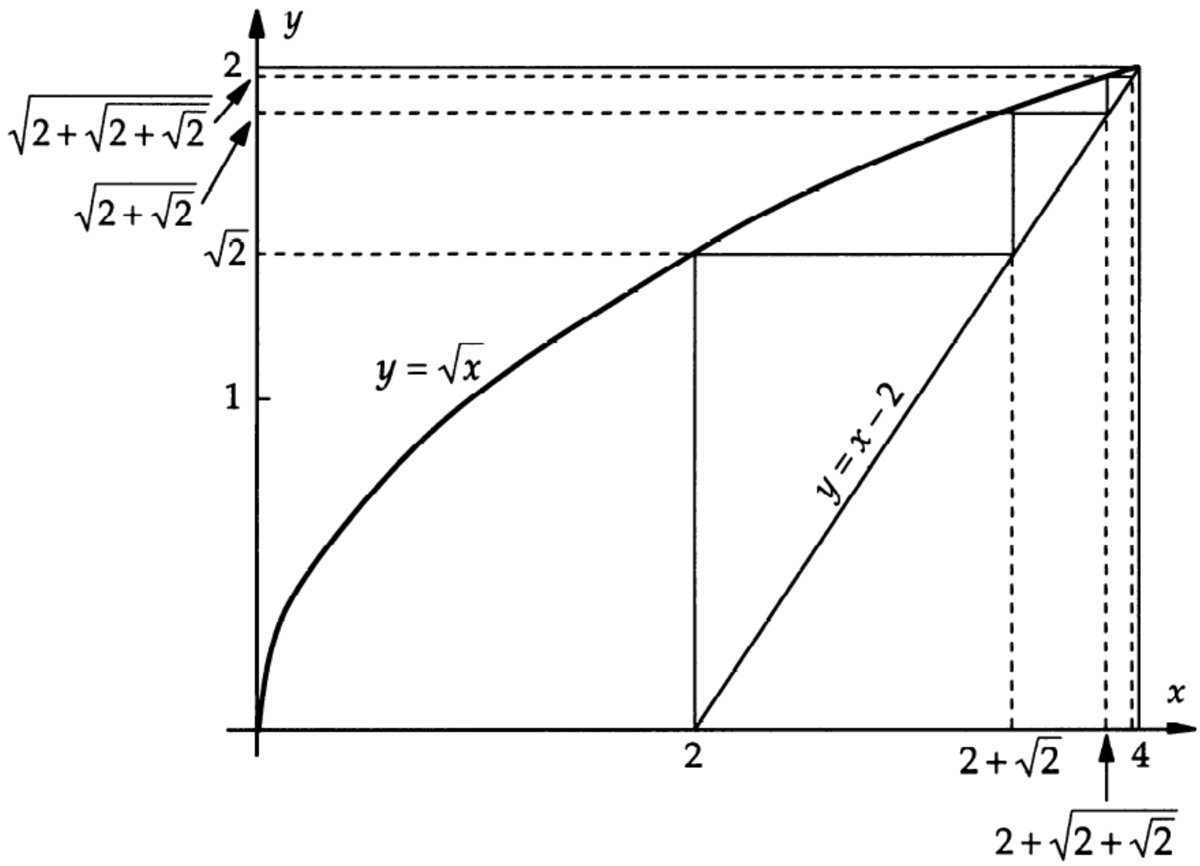




to prove that *the triangle made of medians has three-fourths the area of the original triangle*.

In this activity, you need to be familiar with Cartesian coordinates.

9. Use

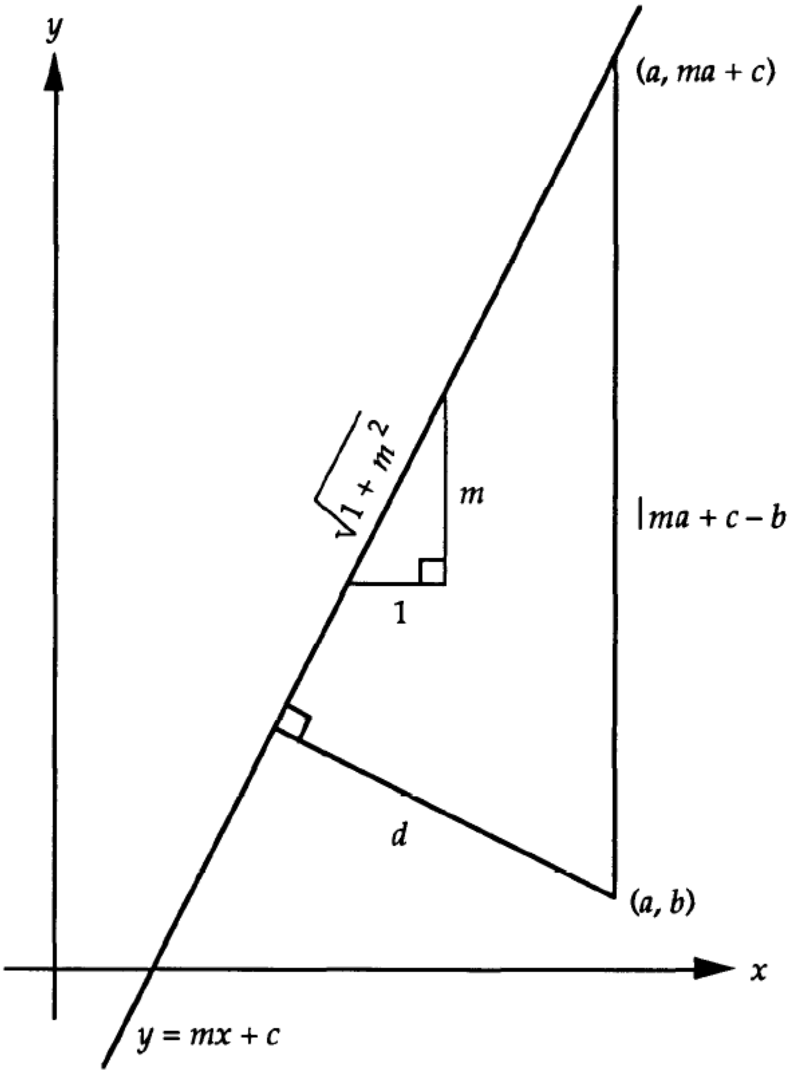


to compute

$$\sqrt{2+\sqrt{2+\sqrt{2+\dots}}} = \ ?$$

In this activity, you need to be familiar with Cartesian coordinates.

10. In



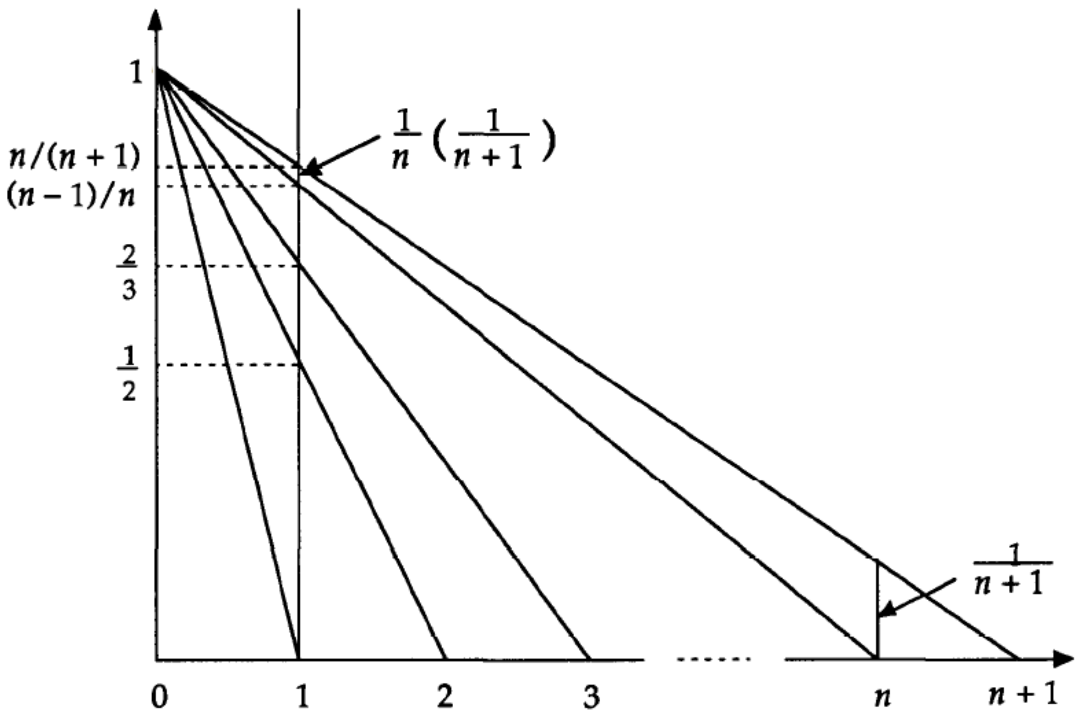
use similarity of triangles to prove

$$d = \frac{|ma + c - b|}{\sqrt{1 + m^2}}.$$

This is the formula for the distance of a point from a line in plane.

In this activity, you need to be familiar with Cartesian coordinates.

11. Use

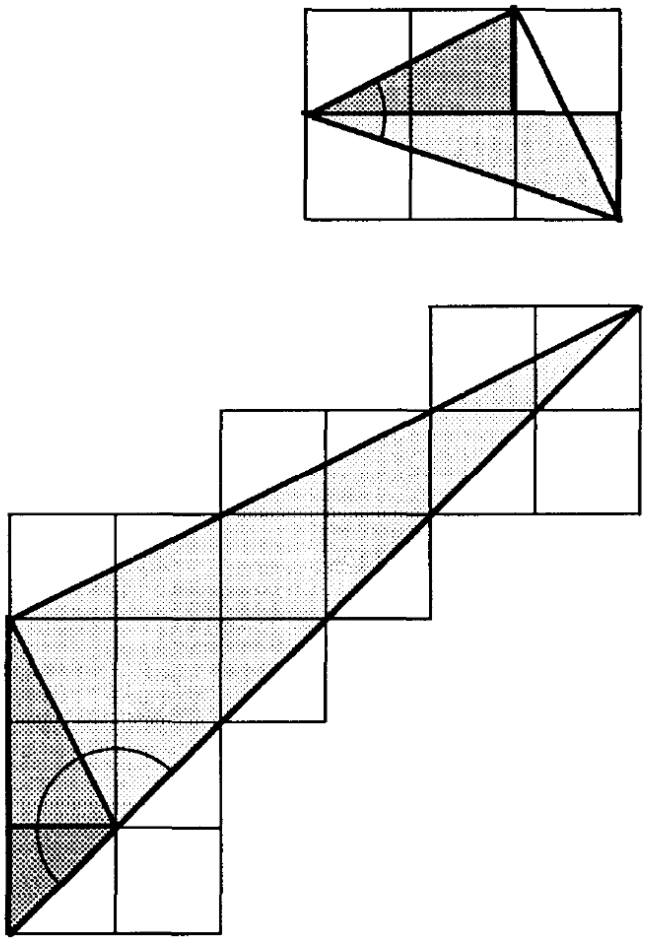


to compute infinite series

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots = \ ?$$

In this activity, you need to know the definition of tangent and its inverse function arctan.

12. Use



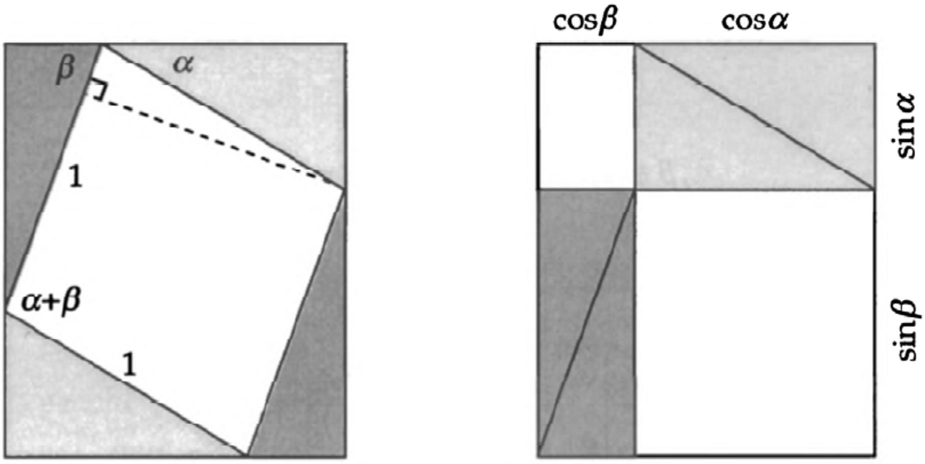
to compute

$$\arctan\frac{1}{2}+\arctan\frac{1}{3} = \ ?$$
$$\arctan 1+\arctan 2+\arctan 3 = \ ?$$

(Answer is given in the last page.)

In this activity, you need to know the definitions of sine and cosine of an angle.

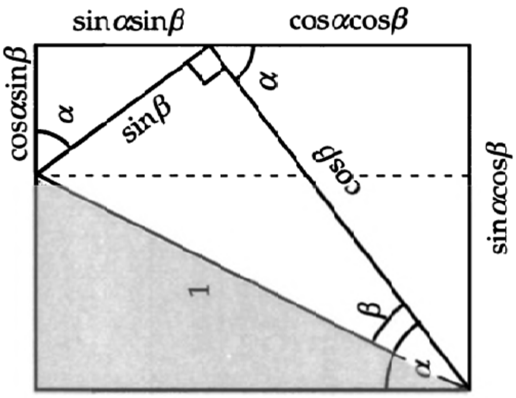
13. In



equating the white area in two figures, prove

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

14. Use



to find formulas for

$$\sin(\alpha-\beta) = \ ?$$
$$\cos(\alpha-\beta) = \ ?$$

Answer for activity 4 is

$$? = \frac{n(n+1)(2n+1)}{6}.$$

Answer for activity 5

$$? = (1+2+3+\cdots+n)^2 = \left(\frac{n(n+1)}{2}\right)^2.$$

Answer for activity 12 are $\frac{\pi}{2}$ and π, respectively.