DINNER PARTIES AND COLORING BOOKS

I. DINING IN CIRCLES

Congratulations! You are in charge of the seating arrangements for $N \geq 3$ guests at a dinner party.

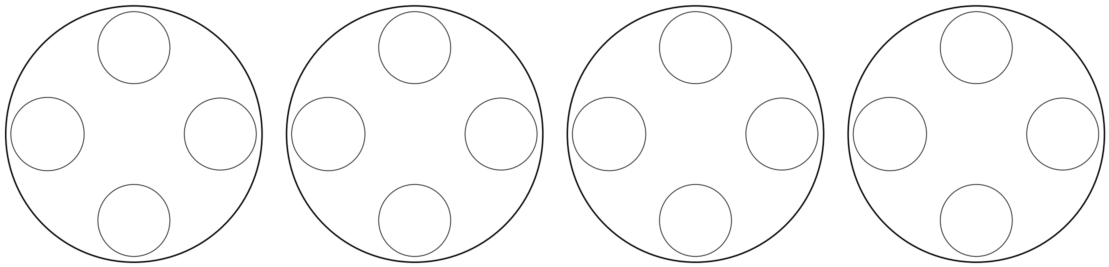
Two tables are set with N equally spaced seats: one for dinner, and one for dessert. Only, the guests all expect *new dining partners on each side* for dessert.

Problem 1. Once you choose the seating for dinner, how many different ways are there to arrange the dessert seat assignments? (Two seat arrangements are considered equivalent if one is a rotation or flip of the other.) This will depend on N: in particular, what is the smallest number of guests with which this is possible?

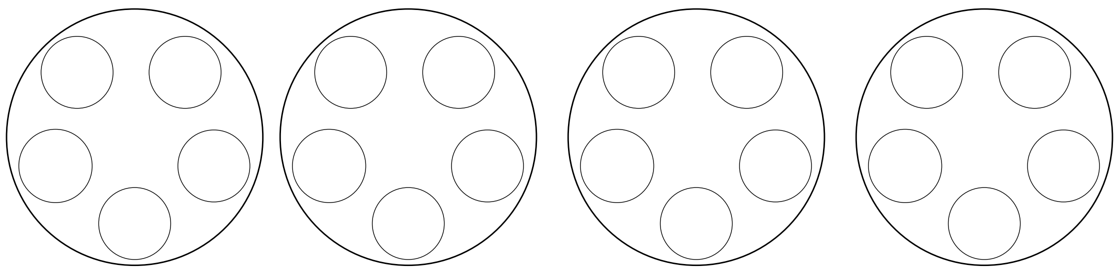
Problem 2. What is the probability, if you just let the guests choose their dessert seats blindfolded, that they all have new dining partners? Try for a few small N, then think about what happens as N gets really big!

[The next page has some tables for you to set for dessert, so you don't have to think about this all in your head. If the guests were numbered 1 to N going around the dinner table, put these numbers in the dessert place settings to describe how you've rearranged their seats.]

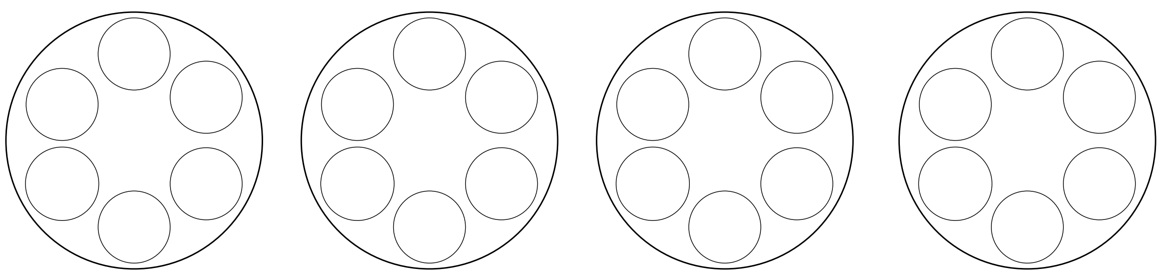
$N = 4$



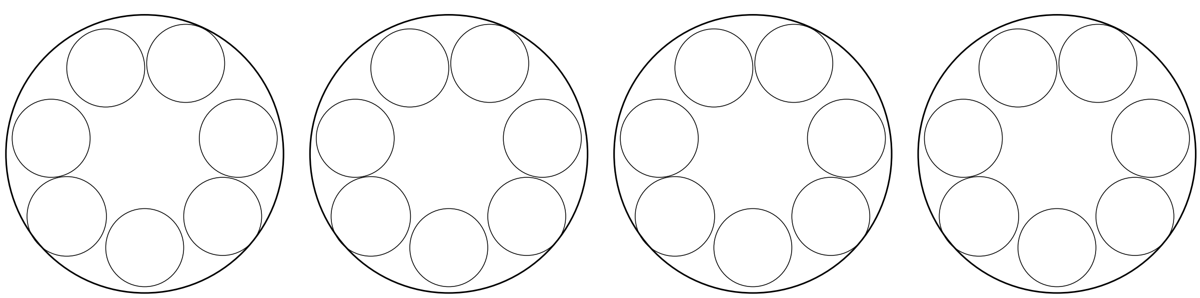
$N = 5$



$N = 6$



$N = 7$

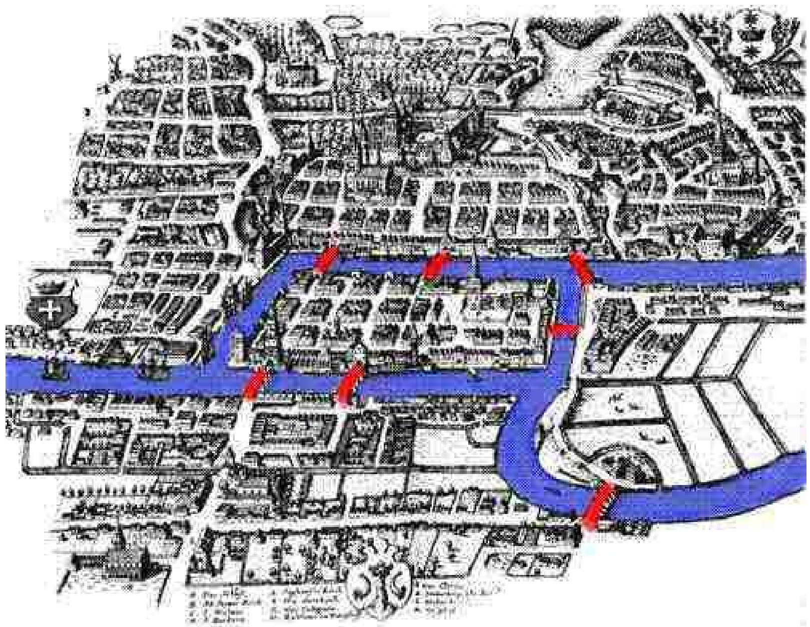


More dinner tables are attached at the end of this packet.

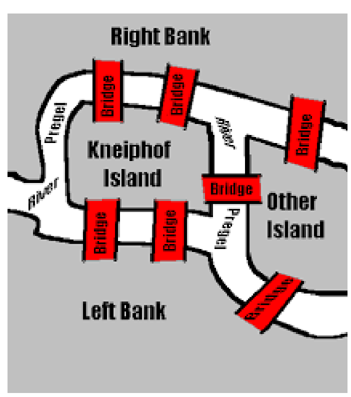
Problem 3. Your first dinner party was such a success that you are again asked to arrange the seats, this time at just one table. This time, each of the $N \geq 3$ guests is a friend of at least half of the other people in the group . . . but they *all* expect to sit between friends! Is this actually possible??

II. WALKING IN CIRCLES

Perhaps you've seen this one before: consider the map of the old Prussian city of Königsberg:



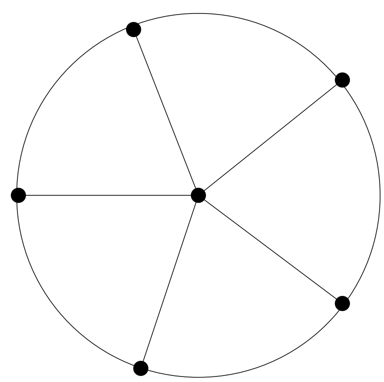
Here's a more schematic picture:

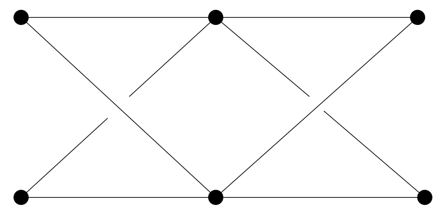


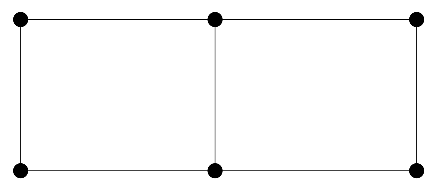
Problem 4. Can you devise a walk through the city that crosses each of the seven bridges exactly once?

III. Circles in Graphs

The solutions to the last two problems are best thought of in terms of *graphs* – which are just sets of *vertices* (points) with *edges* connecting some pairs of (distinct) points:







A *circuit* is a path through a graph that starts and ends at the same vertex. The number of edges emerging from a vertex is the *valence* of that vertex.

In Problem 3, the vertices are guests, and edges connect friends. The problem is now to produce a *Hamiltonian circuit*, i.e. a circuit which uses each vertex exactly once.

In Problem 4, the Königsberg problem, the vertices are land masses, and edges are bridges. Can we find an *Eulerian path*, i.e. a path using each edge exactly once?

Problem 5. Try drawing the Königsberg graph!

Problem 6. Do the graphs on this page have Hamiltonian circuits? Eulerian paths?

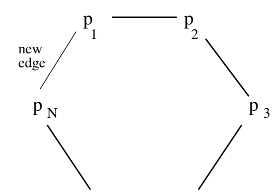
Restating our problems in terms of graphs allows us to think more clearly about them. There are also some powerful tools available:

Euler's Theorem says that if a graph has an Eulerian path, then it has either zero or two vertices of odd valence. (This is true simply because each time an Eulerian path passes through a vertex, it "uses" 2 edges, with the exception of the two endpoints, if they are distinct.) Why does this settle Problem 4?

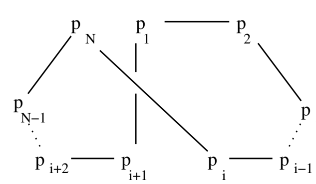
Problem 7. Change the answer to problem 4 by adding one bridge.

Dirac's Theorem says that if a graph has $N \geq 3$ vertices and no "parallel" edges, and each vertex has valence at least $N/2$, then a Hamiltonian circuit exists. Why does this solve Problem 3?

Dirac's theorem is more complicated to justify. Suppose there were graphs with N vertices and the stated properties, but having no Hamiltonian cycle. Among these, pick a graph G which is "maximal" in the sense that adding any edge gives you a graph G' with a Hamiltonian cycle (but the same vertices as G).



Numbering the vertices of G "p_1" thru "p_N" in the order given by this cycle, I claim that for some $2 \leq i \leq N-2$, p_{i+1} is adjacent to p_1 and p_i is adjacent to p_N. Otherwise, there are at least $\frac{N}{2}-1$ vertices p_j (with j chosen from among $2, \ldots, N-2$) adjacent to p_N such that p_{j+1} is not adjacent to p_1. Since p_1 and p_N are also not adjacent to p_1, that means there are at least $\frac{N}{2}+1$ vertices not adjacent to p_1, in contradiction to the valence assumption.

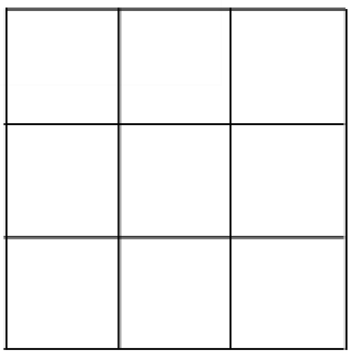


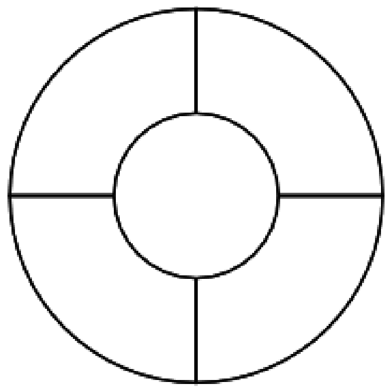
Consequently, $p_1 \to p_2 \to \cdots \to p_i \to p_N \to p_{N-1} \to \cdots \to p_{i+1} \to p_1$ gives a Hamiltonian cycle on G. This contradiction proves Dirac's theorem.

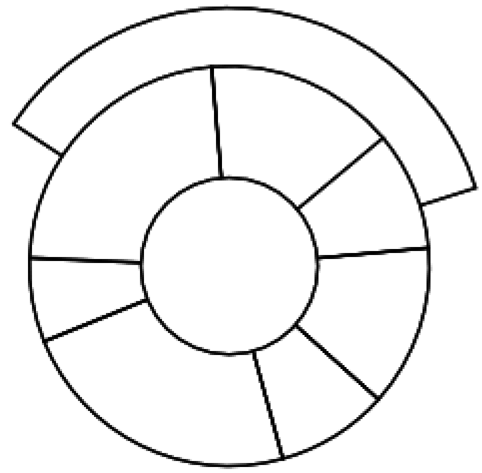
IV. Coloring maps

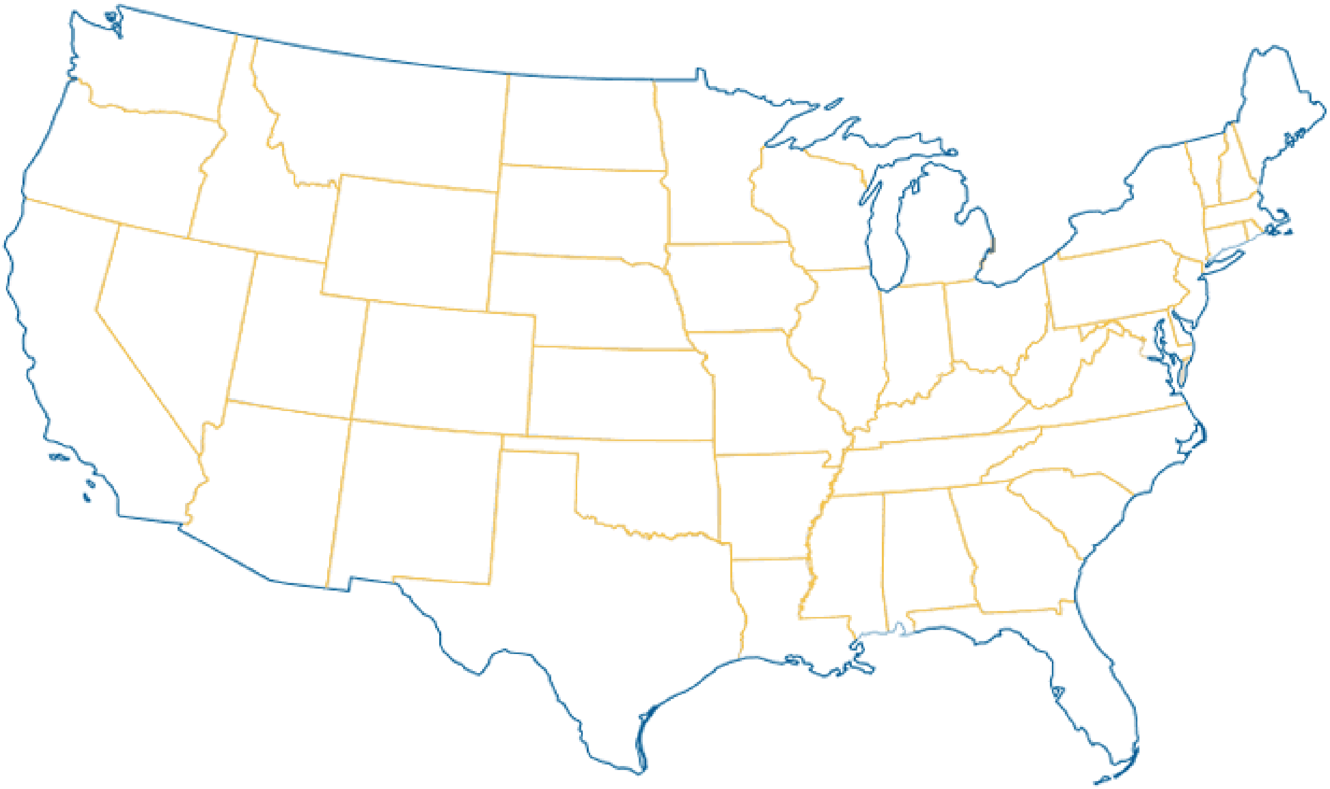
When coloring in a map, you should always use different colors for countries that share a border. (Don't worry about shared corners – those don't count.)

Problem 8. In each of the following maps, how few colors can you get away with?

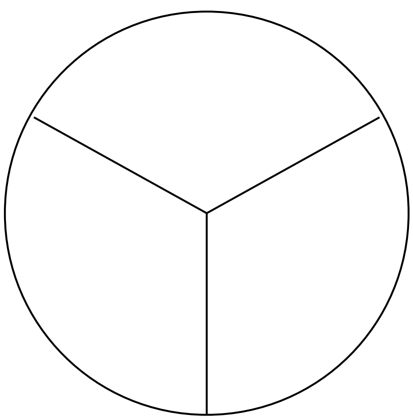


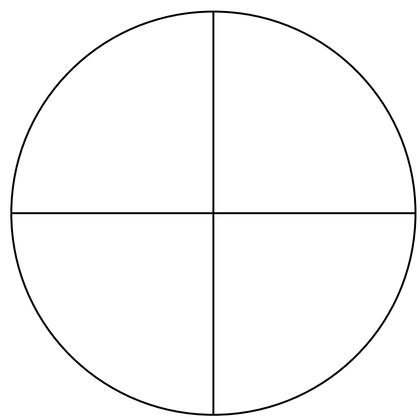






Now look at these rather simpler “maps”:





Problem 9. If adjacent “countries” have distinct colors, how many different ways are there to color these with k colors, for $k = 2$? 3? 4? 5?

Can you find a *formula* for any k?

V. The four-color theorem

This famous result says that every map can be colored with four colors. Many incorrect proofs were given, even *published*, before the correct (but extremely messy and computer-assisted) proof by Kenneth Appel and Wolfgang Haken in 1976. Here we'll just show that six colors always suffice, which is much easier.

Problem 10. For each of the maps in Part IV, draw the *dual graph*: by putting a vertex in each country, joining (only once!) those vertices whose countries share a border, and finally adding a vertex for the "exterior of the map" and edges from it to each country with an external boundary.

Note that each of these graphs can be drawn without any "over-under edge crossings" – we say they are *planar graphs*.

In combinatorics, a k-*coloring* of the graph G is a labeling by $\{1, 2, \ldots, k\}$ of the vertices such that vertices connected by an edge have distinct labels. If G is the dual graph of a map, this is equivalent to coloring the map with k colors (with colors replaced by numbers). In light of Problem 9, it may come as no surprise that there is always a *chromatic polynomial* P_G (of degree equal to the number of vertices of G), such that $P_G(k)$ is the number of k-colorings of G.

So to prove the Six-Color Theorem, we must show that any *planar* graph G has a 6-coloring. (This isn't true for non-planar graphs!) Let E be the number of edges, V the number of vertices, and F the number of regions into which G divides the plane (including the "exterior region" that surrounds the graph). A famous result by Euler says that $F - E + V = 2$.

If there are graphs with no 6-coloring, then let G be one with the smallest number of vertices – a "minimal criminal".

Suppose every vertex of G had valence at least 6. Since each edge contains 2 vertices, we'd then have $2E \geq 6V$, or $V \leq \frac{1}{3}E$.

Each region has at least 3 edges bounding it, and each edge bounds no more than 2 regions: so $2E \geq 3F$, or $F \leq \frac{2}{3}E$.

Using Euler's result,

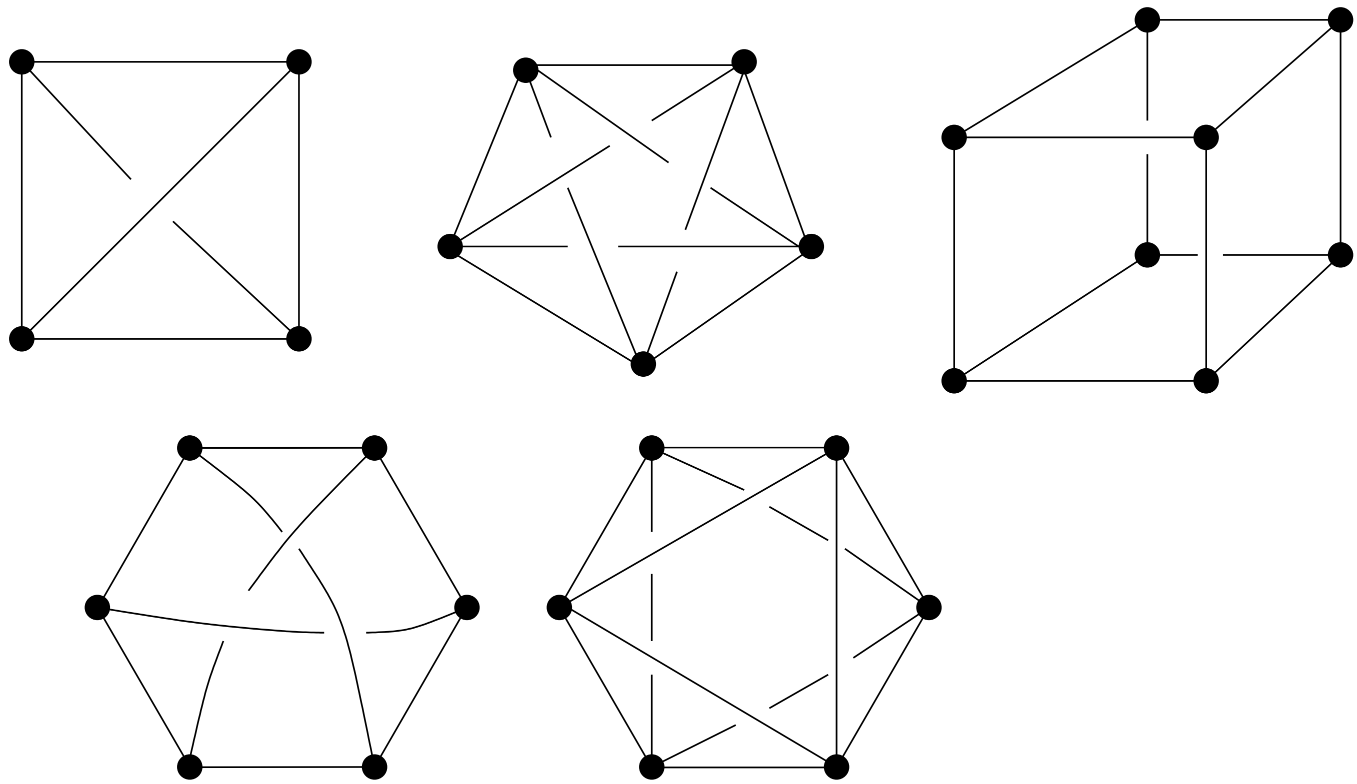
$$2 = F - E + V \leq \frac{2}{3}E - E + \frac{1}{3}E = 0,$$

which is absurd: 2 is bigger than 0!

So there must be a vertex v with valence ≤ 5. Remove it from G. Now you have a graph with fewer vertices, so by "minimality" of G, this new graph has a 6-coloring. But then G itself has a 6-coloring, since v had fewer than 6 neighbors! This contradiction proves the 6-color theorem (for planar graphs, and thus for maps).

VI. (NON-)PLANAR GRAPHS

Here are some graphs with “over-under edge crossings”: that is, they don’t live in the plane the way I’ve drawn them. It might or might not be possible to redraw some edges so that they do.



Problem 11. Which of these graphs is planar, i.e. can be redrawn without crossings?

Which ones have a 4-coloring? How is this related?

VII. Combinatorics

This is the branch of math that deals with counting things – and to which the study of graphs, map-colorings, etc. belong. Here are a few more fun problems about counting stuff. (In some cases using factorial notation $n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$ may be convenient.)

Problem 12. There is an extremely small island in the middle of the Pacific Ocean on which the inhabitants have only four letters in their written language: $\forall$, $\exists$, $\natural$, and $\otimes$. Also, every combination of these letters is a word in their language. How many words do the islanders have consisting of no more than five letters?

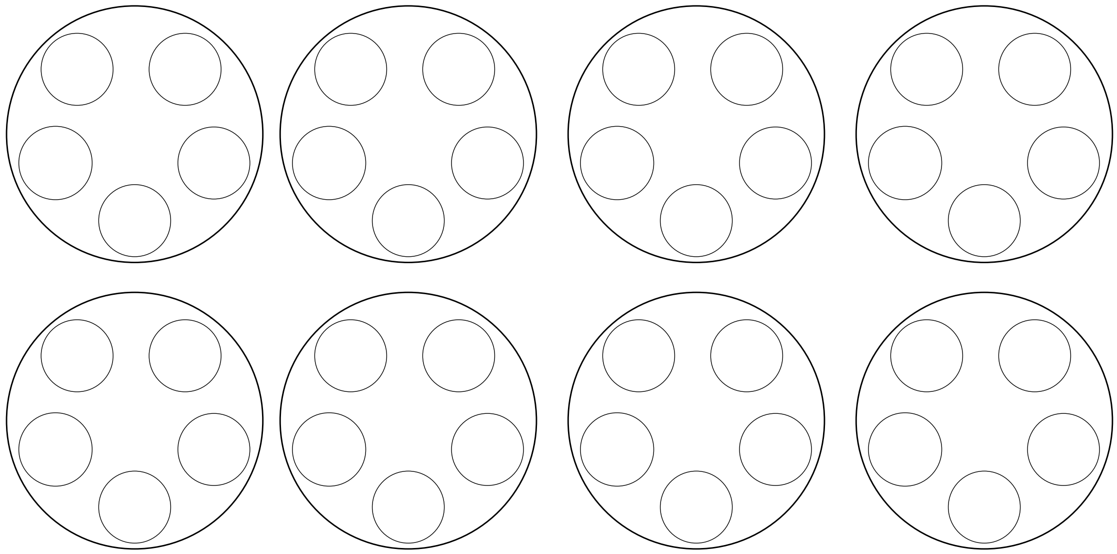
Problem 13. King Arthur has invited all his knights to a feast. How many different ways are there of sitting all 13 knights at their Round Table? As before, two configurations which are rotations of each other are considered to be equal.

Problem 14. Chris and Anna want to trade Pokémon. Chris has 7 fire-type Pokémon, 5 water-type, and 4 grass-type. Anna has 3 fire, 5 water, and 6 grass. They are willing to trade only one Pokémon each, and only for the same type. In how many different ways can they accomplish this?

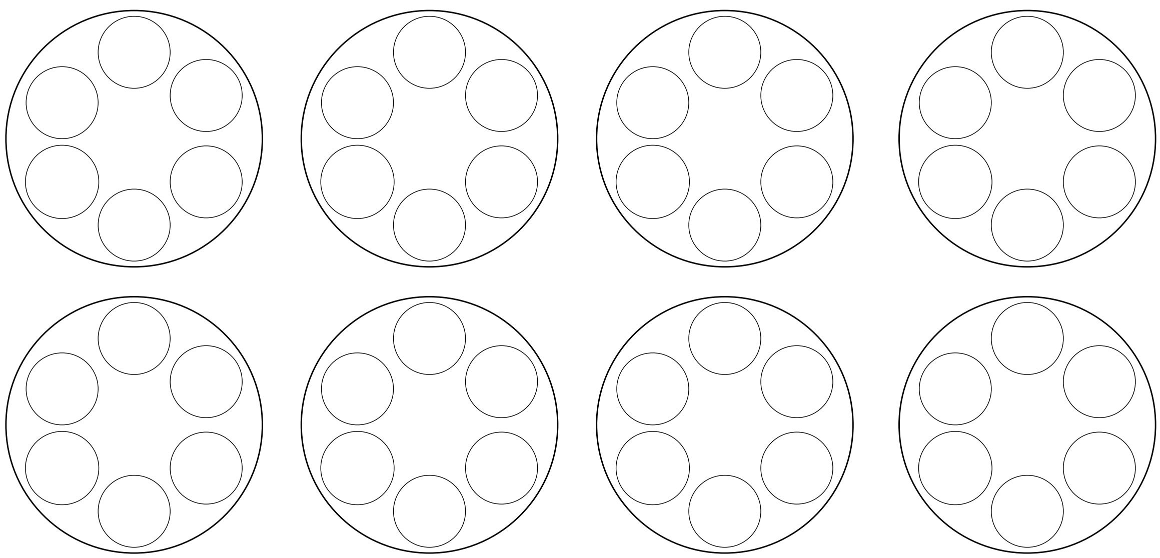
Problem 15. How many different ways can you rearrange the letters in the word MATH? How about CIRCLE? or MATHEMATICS?

Problem 16. A triangulation of a convex $n+2$-gon is a way to cut it into n triangles by drawing some diagonals which do not meet inside the N-gon. Let c_n denote the number of triangulations of an $(n+2)$-gon, called the n^{th} *Catalan number*. Can you find a formula? [Hint: count triangulated $(n+3)$-gons with a marked edge in terms of c_{n+1} and in terms of c_n, to get a formula relating c_{n+1} and c_n.]

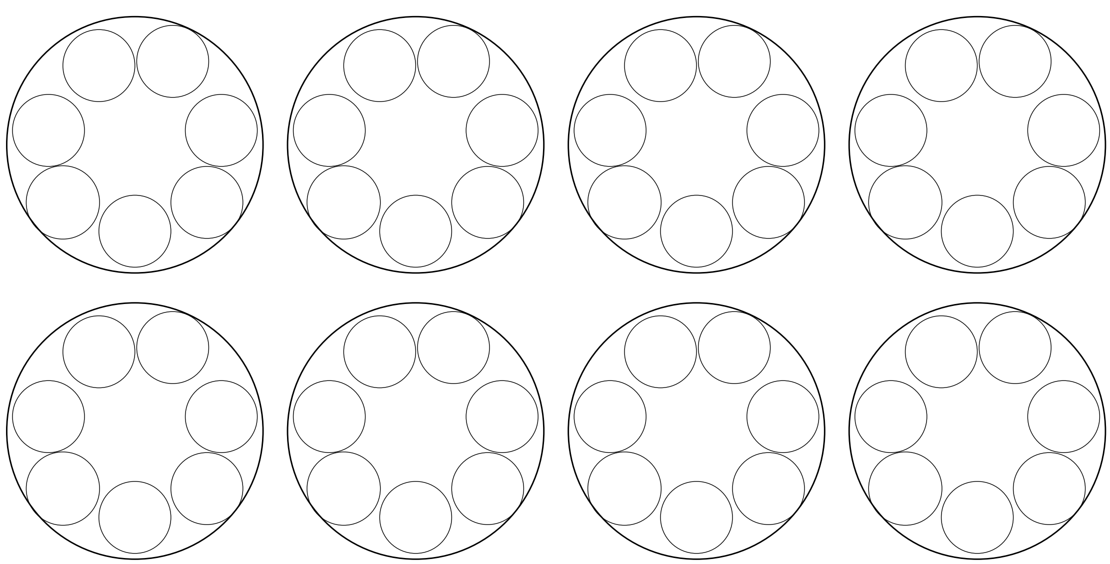
$N = 5$



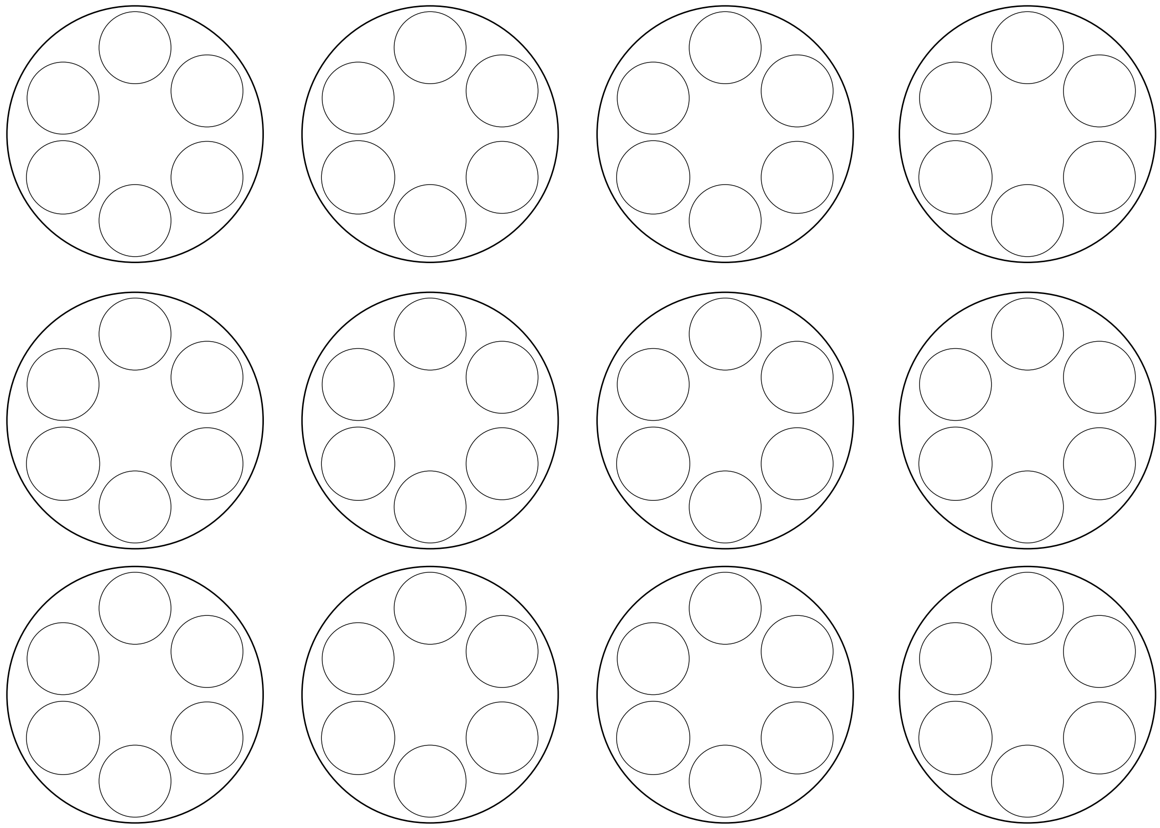
$N = 6$



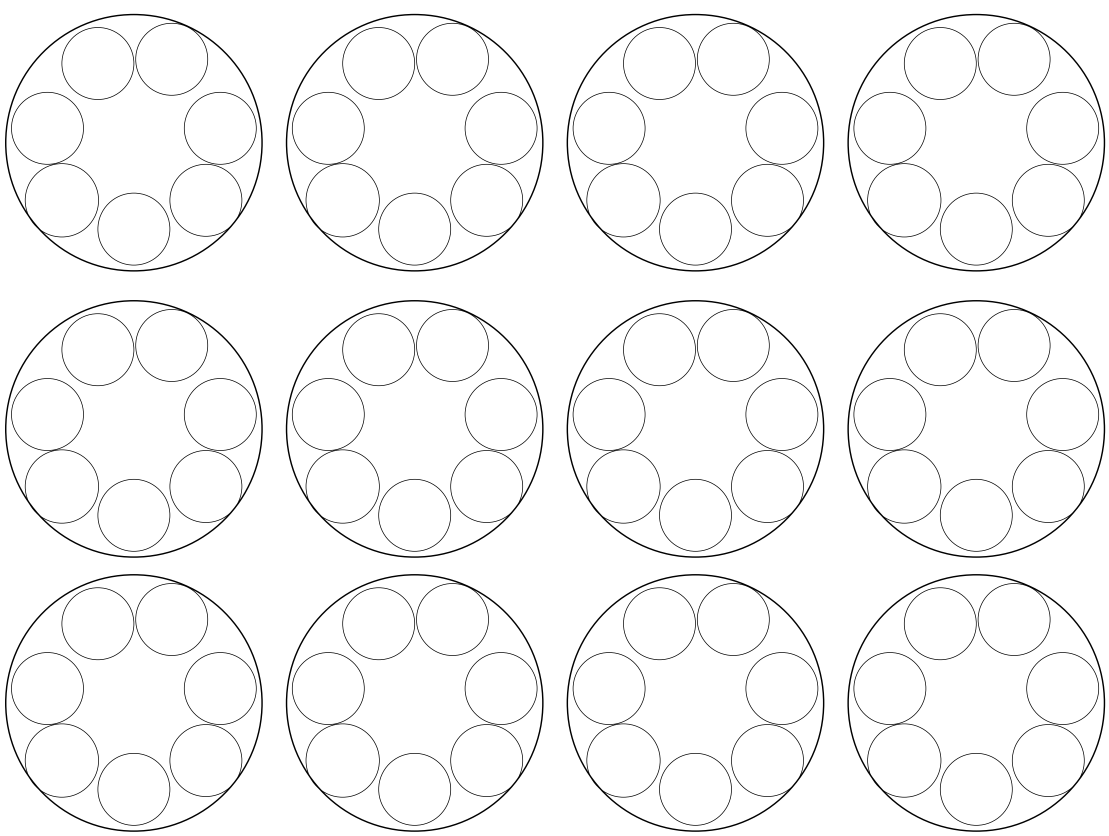
$N = 7$



$N = 6$



$N = 7$



$N = 7$