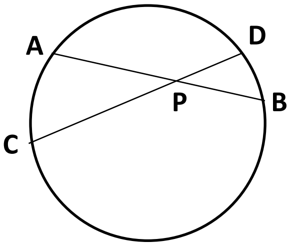
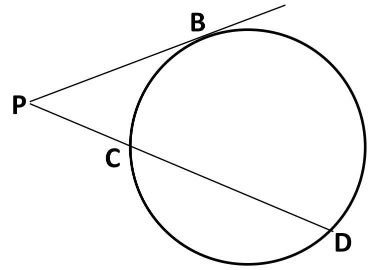
MO-ARML --- September, 2018 -- POWER OF A POINT

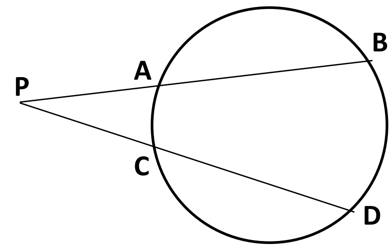
Power of a Point is a set of three-theorems-in-one about circles and line segments.



PA * PB = PC * PD

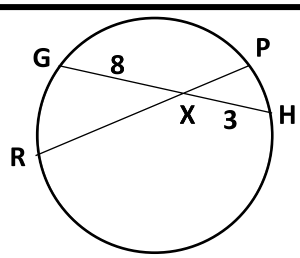


PB^2 = PC * PD

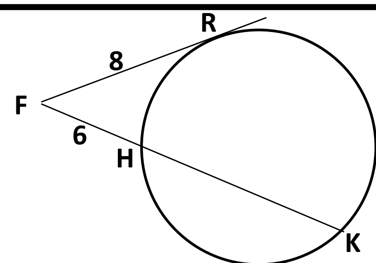


PA * PB = PC * PD

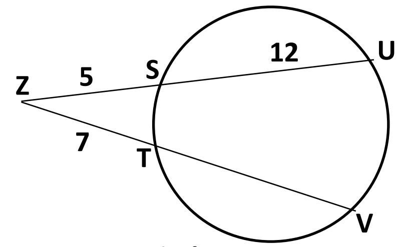
EXERCISES



1A. PR = 14 and PX < RX, find PX.

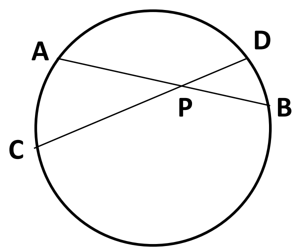


1B. Find HK.



1C. Find TV.

2. **<u>THEOREM</u>: If A, B, C, and D are on a circle then for any point P in the circle, PA * PB = PC * PD. Prove.**

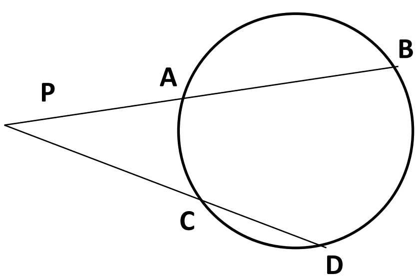


A. Draw AC and BD. Why does ∠CAB = ∠BDC ?

B. Which two triangles are similar? Prove it.

C. Complete the proof that PA * PB = PC * PD .

3. Often other theorems must be used along with Power of a Point. PA = 8; AB = 10; CD = 7; $\angle P = 60^\circ$.

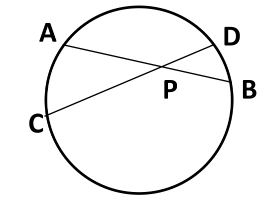


Determine the area of the circle.

AHSME, 1992, #27

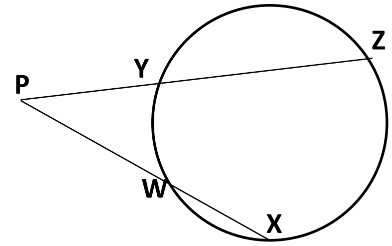
POWER OF A POINT ---- WORKSHEET

1A.



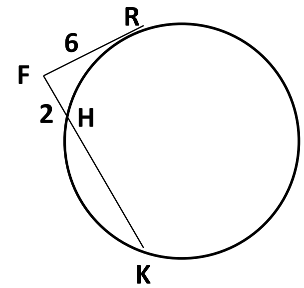
DP = 2; PC = 12; AP = 2PB; find AB

1B.



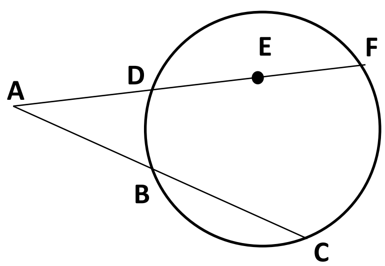
PY = 4; YZ = 8; WX = 2; find PW

1C.



RFK is a right angle.   Find RK.

2. AB = BC and AD = DE = EF. Find AB/AD



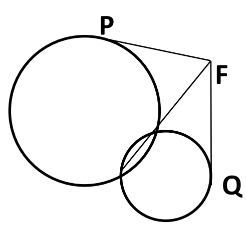
3A. Point P is exterior to circle O with radius r.

A line through P intersects the circle at A and B.

Determine PA*PB in terms of PO and r.

3B. Repeat if P is interior to circle O.

4.



Two circles intersect at points A and B. F is on line AB such that FP and FQ are tangents to the circle.   <u>PROVE</u>:   FP = PQ.

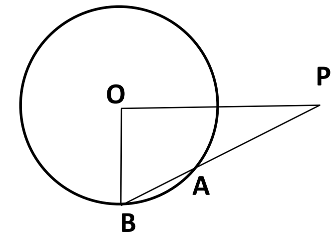
5A. Point P is exterior to a circle with PA as a tangent. A line through P intersects the circle at points B and C.
<u>PROVE</u>: **PA^2 = PB * PC**

1. Draw segments AB and AC. Why does ∠ACP = ∠BAP ?
2. Which two triangles are similar? Why?
3. Complete the proof.

5B. A second line through P intersects the circle at points D and E.   PROVE the Corollary:

PB * PC = PD * PE   *[Being a 'corollary' suggests that there is a very easy, short proof!]*

6. PO is perpendicular to OB and PO equals the length of the Diameter of circle O. Compute PA/AB.



7. A and B are points on a circle with center O. C lies outside the circle, on ray $\overrightarrow{AB}$. Given AB = 24, BC = 28, and OA = 15, find OC.

   Hint: The solution to a “recent” problem is helpful here.

8. Equilateral triangle ABC is inscribed in a circle. Q is on segment BC. P is on line AQ and on the circle.

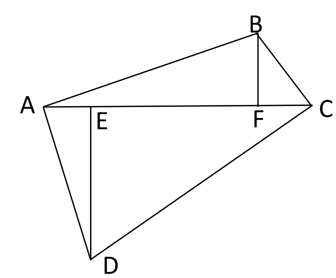
   PROVE: $\frac{1}{PQ} = \frac{1}{PB} + \frac{1}{PC}$

9. An equilateral triangle ABC is inscribed in a circle. D and E are midpoints of AB and BC respectively. F is the point where ray $\overrightarrow{DE}$ intersects the circle. Compute DE/EF.

10. In the figure, ABCD is a quadrilateral with right angles at A and C.

    E and F are on AC such that DE and BF are perpendicular to AC.

    If AE = 3, DE = 5, and CE = 7, compute BF.



ANSWERS with HINTS of solutions

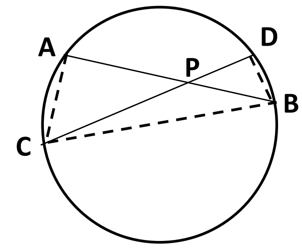
EXERCISES

1A. XG*XH = XR*XP Let x = PX, then XR = 14 – x; 8*3 = x(14 – x) → $x^2 - 14x + 24 = (x - 12)(x - 2) = 0$. X = 2 or 12

Since PX < RX, PX = **2**.

1B. FR^2 = FH * FK, Let HK = x, then FK = x + 6; 8*8 = 6 * (x + 6); x = **14/3**

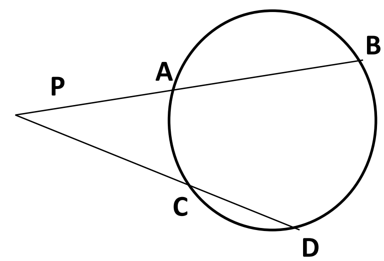
1C. ZS * ZU = ZT * ZV; Let TV = x, then ZV = 7 + x. 5 * 17 = 7 * (7 + x) x = **36/7**



2A. Draw BC. Angle A and angle D each subtend the same chord BC.

2B. Since angle A is congruent to angle D and since the vertical angles at P are equal, by AA, ΔPAC is similar to ΔPDB

2C. Use ratios generated by these similar triangles: PA/PD = PC/PB or PA* PB = PC * PD.



3. PA * PB = PC * PD. Let CD = x, then PD = PC + CD = PC + 7; 8 * 18 = PC * (PC + 7). Solving: PC = 9.

Since $\angle P = 60^\circ$, PC = 9, and PB = 18, PCB is a 30-60-90 triangle with right angle C.

Therefore, $BC = 9\sqrt{3}$. And $\angle$BCD is also a right angle.

So, $BD = \sqrt{81 * 3 + 49} = \sqrt{292} = 2\sqrt{73}$. Since $\angle$BCD is a right angle, BD is a diameter of the circle and it radius is $\sqrt{73}$. The area of the circle = πr^2 = **73 π**

PROBLEMS

1A. Let PB = x, then AP = 2x. PA*PB = PC*PD or 2x * x = 2 * 12; $2x^2 = 24$; $x = 2\sqrt{3}$; AB = 3x = $\mathbf{2\sqrt{3}}$

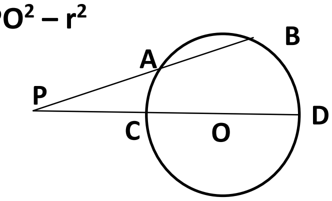
1B. Let PW = x, then PX = x + 2. PY*PZ = PW*PX. 4*12 = x(x + 2); $x^2 + 2x - 48 = 0 = (x + 8)(x - 6)$. PW = **6**

1C. Let HK = x. FR^2 = FH * FK. $6^2 = 2(x + 2)$. X = 16 and FK = 18. RK = $\sqrt{6^2 + 18^2} = \sqrt{360} = \mathbf{6\sqrt{10}}$

2. Let x = AD = DE = EF; y = AB = BC. AD * AF = AB * AC. x * 3x = y * 2y; $3x^2 = 2y^2$; AB/AD = y/x = $\sqrt{3/2}$ or $\sqrt{6}/2$

3A. Let radius = r, then PC = PO – r; and PD = PO + r; PA * PB = PC * PD = (PO – r)(PO + r) = $\mathbf{PO^2 - r^2}$

3B. A similar solution yields: PA * PB = $\mathbf{r^2 - PO^2}$

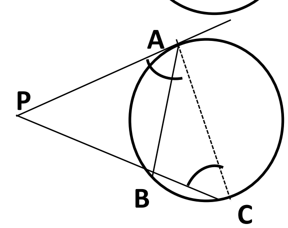


4. FP^2 = FA * FB but FQ^2 also equals FA * FB; hence, FP = FQ.

5A. 1. See diagram. For a tangent PA and chord AB, the *Alternate Segment Theorem* states that for any point C on 'alternate' side of circle, $\angle$PAB = $\angle$ACB

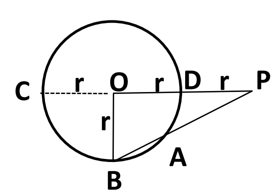
2. Triangles APB and CPA by the AA theorem for similar triangles.

3. Use ratios generated by these similar triangles.



5B. By Power of Point, PA^2 = PB * PC and PA^2 = PD * PE; hence, PB * PC = PD * PE

6. By Pythagorean Theorem, PB = $r\sqrt{5}$. PC * PD = PA * PB. 3r*r = PA* $r\sqrt{5}$

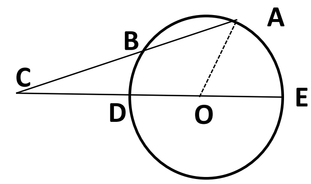


PA = $\frac{3r}{\sqrt{5}} = \frac{3r\sqrt{5}}{5}$; AB = PB – PA = PA/AB = $r\sqrt{5} - \frac{3r\sqrt{5}}{5} = \frac{2r\sqrt{5}}{5}$

PA/AB = $\frac{3r\sqrt{5}}{5} / \frac{2r\sqrt{5}}{5}$ = **3/2** .

7. From 3A, above, CB * CA = $CO^2 - r^2$, 28 * 52 = $OC^2 - 15^2$

OC^2 = 1456 + 225 = 1681;   OC = **<u>41</u>**



8. Since no dimensions are given, WLOG let AB = BC = AC = 1.   Let CQ = d, BQ = 1 – d, PB = x, and PC = y.

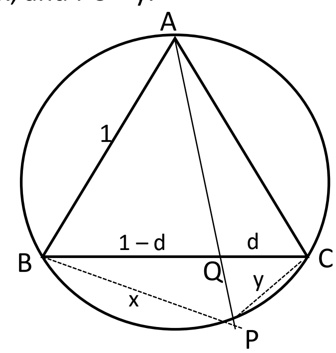
We must show that: $\frac{\mathbf{1}}{\mathbf{PQ}} = \frac{\mathbf{1}}{\mathbf{x}} + \frac{\mathbf{1}}{\mathbf{y}}$. Via Similar triangles, we will "hunt for" 1/x and 1/y.

Angles BAP and BCP are congruent -- each subtends BP. With angle Q shared,

ΔAQB ≅ ΔCQP.   AB/CP = BQ/PQ; **1/y = (1 – d)/PQ** .

Similarly, angles PBC and PAC are congruent and ΔACQ ≅ ΔBPQ.

AC/BP = CQ/PQ;   **1/x = d/PQ.**   Therefore, $\frac{1}{PB} + \frac{1}{PC} = \frac{1}{x} + \frac{1}{y} = \frac{d}{PQ} + \frac{1-d}{PQ} = \frac{1}{PQ}$ QED

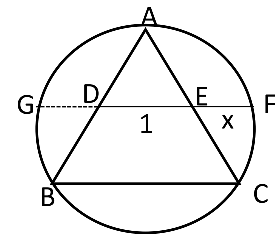


9. WLOG, let AB = BC = CA = 2, then AD = DE = EA = 1. Let EF = x = DG.

Using Power of Point on E: EA * EC = EG * EF or 1*1 = (x+1) * x.

So: $x^2 + x - 1 = 0$.   By quadratic formula, x = $(-1 + \sqrt{5})/2$.

DE/EF = 1/x = = $(1 + \sqrt{5})/2$, the Golden Ratio.



10. AHSME, 1990, #20   Since A and C are right angles, they are each on the circle with diameter BD. Therefore, A, B, C, and D are cyclic and Power of Point can be applied. Extend DE to X on the circle. Since X is on the circle with BD diameter, BDX is a right triangle and DXB is a right angle. Thus BFEX is a rectangle with BF = EX. Apply Power of Point to E.

EX * ED = EA * EC   BF = EX = 3 * 7/5 = **<u>21/5.</u>**