Fantastic Factoring

Following are some factoring patterns that you might already recognize. *x* and *y* can both represent variables in the expressions, or *y* might be a constant. These rules work for all real numbers *x* and *y*. Sometimes you are given the factored form and recognizing the pattern will save you time and errors in not multiplying out all of the terms.

Difference of Squares

$x^2 - y^2 = (x-y)(x+y)$

Binomial Squares

$x^2 - 2xy + y^2 = (x-y)(x-y)$

$x^2 + 2xy + y^2 = (x+y)(x+y)$

Difference of Cubes

$x^3 - y^3 = (x-y)\left(x^2 + xy + y^2\right)$

Sum of Cubes

$x^3 + y^3 = (x+y)\left(x^2 - xy + y^2\right)$

Factor the following expressions.

1. $64a^4 - 100b^4$
2. $27m^3 + 8$
3. $4b^2 - 36bc + 81c^2$

By looking at the sum and difference of cubes above, how do you think the general difference and sum below would be factored?

$x^n - y^n =$

For all odd *n*, $x^n + y^n =$

Why does the sum only work for odd *n*?

Often an expression is not a binomial or trinomial and has 4 or more terms. We use different methods to factor these. For example, let's use **factoring by grouping** to simplify and solve.

4. $4ab - 8b^2 + 3a^3 - 6a^2b$
5. $xy + y + x + 1 = 0$

The entire purpose of factoring a polynomial is to help in simplifying and solving polynomial equations. Most problems you have probably seen have set the equation equal to 0 to solve; however, if looking for integer solutions, this doesn't always have to be the case.

6. If x is a positive integer and $x(x+1)(x+2)(x+3)+1=379^2$, compute x.

What if factoring by grouping doesn't work? **Simon's Favorite Factoring Trick** to the Rescue! SFFT allows you to think about the problem algebraically or visually by completing the rectangle.

Example: Given that j and k are integers and $j^2+5j^2k^2-20k^2=109$, find $5j^2k^2$.

7. Both p and q are positive integers where $p>q$. Find all ordered pairs (p, q) such that $2pq+2p-3q=18$.

8. Twice the area of a non-square rectangle equals triple its perimeter. If the dimensions are both positive integers, what is the rectangle's area?

9. Compute all integer values of n, $90 \le n \le 100$, that can not be written in the form $n = a + b + ab$, where a and b are positive integers.

10. Compute the positive integer x such that $4x^3 - 41x^2 + 10x = 1989$.

11. If $x^5 + 5x^4 + 10x^3 + 10x^2 - 5x + 1 = 10$ and $x \neq -1$, compute the numerical value of $(x+1)^4$.

12. Let A, M, and C be nonnegative integers such that $A + M + C = 12$. What is the maximum value of $A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A$? (Hint: Look back at question #5 and its solution.)

13. Find the number of ordered pairs of integers (m, n) for which $mn \geq 0$ and $m^3 + n^3 + 99mn = 33^3$ is true.

14. Let us examine the expression $a^3 + b^3$, where $a > b$. One well-known result is that $a^3 + b^3 = c^3$ has no solution in positive integers. For each of the equations below, either:

1. Prove that no solutions can exist OR
2. Show how an infinite number of solutions can be generated.

A. $a^3 + b^3 = c^2$

B. $a^3 + b^3 = c \cdot d \cdot e$, where c, d, and e are in geometric progression

C. $a^3 + b^3 = c \cdot d \cdot e$, where c, d, and e are in arithmetic progression

D. $a^3 + b^3 = 3p$, where p is a prime greater than 3

Solutions to Fantastic Factoring

1. $4(4a^2-5b^2)(4a^2+5b^2)$
2. $(3m+2)(9m^2-6m+4)$
3. $(2b-9c)^2$
4. $(a-2b)(4b+3a^2)$
5. $x = -1$ *or* $y = -1$
6. 18 (1989 ARML, Individual #1)
7. (4, 2)
8. 48
9. 96 and 100 (1990 ARML, Team #7)
10. 13 (1989 NYSML, Individual #2)
11. 10 (1994 ARML, Team #1)
12. 112 (2000 AMC, #12)
13. 35 (1999 AHSME, #30)
14. A. Infinite number of solutions
    B. No solutions
    C. Infinite number of solutions
    D. No solutions
    (1990 ARML PQ Part I)

Team Round Answers

1. 6481 (1992 NYSML, Team #5)
2. 186 (mathleague.org 11207, Large Team #4)
3. 2013 (mathleague.org 11607, Team #2)
4. 6 (mathleague.org 11301, Sprint #10)
5. $\pm 3i$ (1991 NYSML, Individual #2)
6. 1600 (mathleague.org 11202, Large Team #7)
7. -61 (AHSME 1966, #30)
8. -403 (mathleague.org 11607, Target #6)
9. 4 (mathleague.org 11308, Sprint #28)
10. 96 (mathleague.org 11307, Sprint #11)

Team Round

30 minutes - 10 questions - maximum of 6 team members

There is no penalty for guessing.

1. The number $(9^6 + 1)$ is the product of three primes. Compute the largest of these primes.

2. Of the integers between 1 and 2310, how many are divisible by exactly three of the five primes 2, 3, 5, 7, and 11?

3. If x and y are positive integers such that $x^2 = y^2 + 61$, find $x(x+2) + y(y+3)$.

4. The graph of $xy + 3x + 2y = 0$ can be produced by translating the graph of $y = \frac{k}{x}$ to the left and down for some constant value k. Find k.

5. Let $f(x) = x^2 + bx + 9$ and $g(x) = x^2 + dx + e$. If $f(x) = 0$ has roots r and s, and $g(x) = 0$ has roots $-r$ and $-s$, compute the two roots of $f(x) + g(x) = 0$.

6. How many ordered pairs of integers (x, y) with $1 \le x \le 100$ and $1 \le y \le 100$ make the quantity $xy - x - y$ a multiple of 5?

7. If three of the roots of $x^4 + ax^2 + bx + c = 0$ are 1, 2, and 3, find the value of $a + c$.

8. x and y are real numbers that satisfy the equations $x - y = 1$ and $x^5 - y^5 = 2016$. Find $\frac{x^5+y^5}{x+y} - (x^4 + y^4)$.

9. How many ordered pairs of positive integers (a, b) are there such that $\frac{1}{a} - \frac{1}{b} = \frac{1}{143}$?

10. Suppose that a, b, c, d are real numbers such that $ab + 3a + 3b = 216$, $bc + 3b + 3c = 96$, $cd + 3c + 3d = 40$. Find the maximum possible value of $ad + 3a + 3d$.