WUSTL Math Circle

Approximating π by Experiment

We are going to approximate the most important constant in mathematics

$$\pi$$

which is the ratio of a circle's circumference to its diameter. It is approximately

$$\pi \approx 3.14.$$

In more decimals

$$\pi \approx 3.14159265358979323846264338327950288419716939937510.$$

Here are the first one thousand digits of π.

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3.14159265358979323846264338327950288419716939937510
  58209749445923078164062862089986280348253421170679
  82148086513282306647093844609550582231725359408128
  48111745028410270193852110555964462294895493038196
  44288109756659334461284756482337867831652712019091
  45648566923460348610454326648213393607260249141273
  72458700660631558817488152092096282925409171536436
  78925903600113305305488204665213841469519415116094
  33057270365759591953092186117381932611793105118548
  07446237996274956735188575272489122793818301194912
  98336733624406566430860213949463952247371907021798
  60943702770539217176293176752384674818467669405132
  00056812714526356082778577134275778960917363717872
  14684409012249534301465495853710507922796892589235
  42019956112129021960864034418159813629774771309960
  51870721134999999837297804995105973173281609631859
  50244594553469083026425223082533446850352619311881
  71010003137838752886587533208381420617177669147303
  59825349042875546873115956286388235378759375195778
  18577805321712268066130019278766111959092164201989
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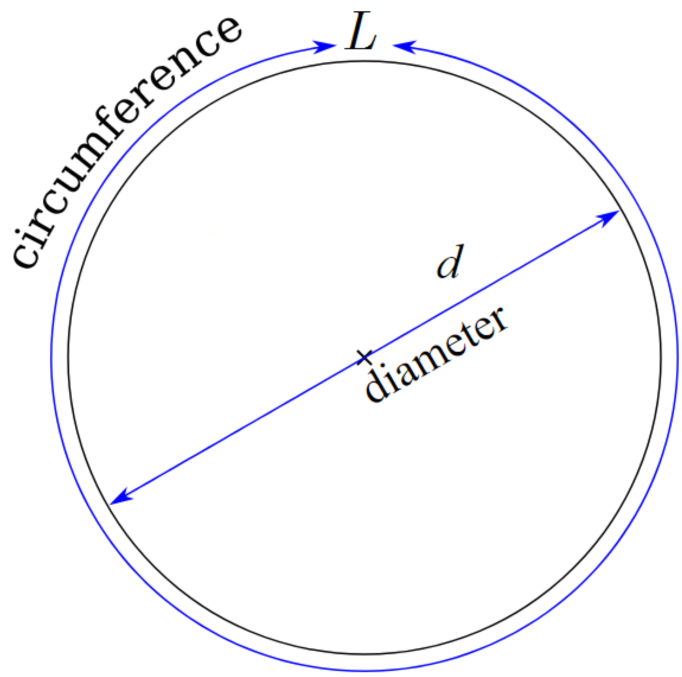
This decimal expansion never ends and has no repeating pattern. Today we use different physical experiments to approximate π.

1. By definition, π is the ratio of a circle's circumference to its diameter. This suggests the following experiment to approximate π.

   Choose a cotton thread, of whatever length. Measure its length L. Then make the best circle you can make out of your thread, and measure its diameter d. Now you can approximate π by

$$\pi \approx \frac{L}{d}.$$



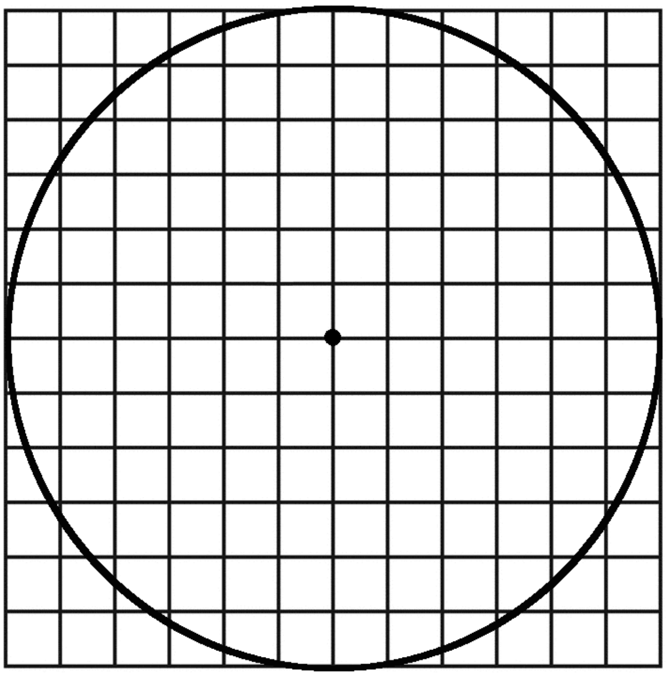
2. Do this experiment with several different threads. Record your estimations for π. Find the average of your records.

3. Another very common definition of π is the ratio of a circle's area to the square of its radius. This suggests the following experiment to approximate π.

   I give you a graph paper. You decide on a lattice point, and by a compass draw a circle centered at your lattice point and radius some integer unit. Count the number N of unit squares inside your circle. Now you can approximate π by

$$\pi \approx \frac{N}{r^2}.$$

   For example in the following figure, my radius is 6 units, and there are approximately 13 unit squares inside the circle. This gives the approximation $\pi \approx \frac{112}{(6)^2} = 3.1$.

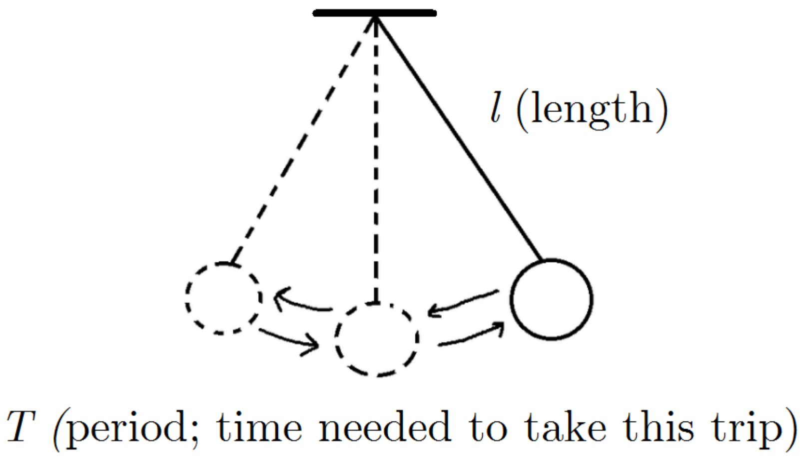


4. Try to do it with several different circles. Record your estimations for π. The larger your radius, and the more refined scales you use, the better becomes your estimation. Find the average of your records.

A pendulum is a weight suspended from a pivot so that it can swing freely. When a pendulum is displaced sideways from its resting, it oscillates about the equilibrium position, swinging back and forth. The time for one complete cycle, a left swing and a right swing, is called the *period*. For small initial displacements, the period T of a pendulum of length l is given by

$$T = 2\pi\sqrt{\frac{l}{g}}, \tag{1}$$

where g is the Earth's gravitational acceleration.



5. Let us first approximate g. To do this, release a small mass from a height above the ground, and measure the time it takes for the mass to hit the ground. Then g is given by

$$g = \frac{2 \times \text{height}}{(\text{hitting time})^2}.$$

(It does not matter what units you use for measuring height ot time; but whatever unit you decide on, use it for the rest of your experiments with pendulum.) Use this experiment several times, and record your estimates for g. Find their average.

6. Now we build a large pendulum in the room hanging from the ceiling. We measure its length and period. Now put all these values together with your estimation for g (from the previous activity) in formula (1) to estimate π:

$$\pi \approx \frac{\text{period}}{2}\sqrt{\frac{g}{\text{length}}}.$$

7. It can be shown that the probability that two randomly chosen integers have no common prime factor is $\frac{6}{\pi^2}$.[1] This suggests the following experiment to approximate π.

   I ask each of you to choose two random positive integers, whatever numbers you like. Then you should figure out whether your pair have common prime factors or not. You can do this by hand or use https://www.wolframalpha.com/, and enter the following question in its command line:

   Are YOUR FIRST NUMBER and YOUR SECOND NUMBER relatively prime?

8. If we have N pairs of integers, and M of our pairs have no common prime factor, then

$$\pi \approx \sqrt{\frac{6N}{M}}.$$

[1] An elementary proof you can find in the book *Challenging Mathematical Problems with Elementary Solutions*, A. M. Yaglom and I. M. Yaglom, Dover Publications, 1964, Volume I, Problems 92-3.

9. It can be shown that[2]:

   > If a short needle, of length l, is dropped on paper that is ruled with equally spaced lines of distance $d \geqslant l$, then the probability that the needle comes to lie in a position where it crosses one of the lines is exactly $\frac{2l}{\pi d}$.

   This suggests the following experiment to approximate π.

   I give you a single lined paper and a small wood stick. You throw the stick on the paper several times (the more the better), count the number of times the stick crosses the lines. This gives you the approximation

$$\pi \approx \frac{2 \times \text{stick length} \times \text{number of throws}}{\text{distance between lines} \times \text{number of crossings}}.$$

[2]Several proofs for this fact you can find in the book *Proofs from the Book*, M. Aigner and G. Ziegler, Springer Verlag, 2010, Edition 4, Chapter 24.

10. A challenging problem for those of you of higher education is to justify the formulas in activity 9, or even 7.