MO-ARML --- January, 2019 -- AMC 12 Practice

The AMC 10A and AMC 10B are 25-question tests offered each February by the Mathematical Association of America. The time limit is 75 minutes and calculators are NOT allowed. Each question is multiple choice with 5 options. The scoring is: *Correct answer: 6 points; No answer: 1.5 points; Incorrect answer: 0 points.* With this scoring, students should NOT *'just guess'* any answers.

More information can be found at: https://www.maa.org/math-competitions

Past tests and solutions are at: https://artofproblemsolving.com/wiki/index.php/AMC_Problems_and_Solutions .

These 16 questions are selected from #11 – 20 on the 1997 AMC and 2010 12B tests.

1. In the 6th through 9th basketball games of a season, a player scored 23, 14, 11, and 20 points. Her points per game average was higher after 9 games than it had been after 5 games. If her average after 10 games was greater than 18, what is the least number of points that she could have scored in the 10th game?

   A. 26 B. 27 C. 28 D. 29 E. 30

2. If m and b are real numbers and $mb > 0$, then the line whose equation is $y = mx + b$ cannot contain the point:

   A. (0, 1997) B. (0, – 1997) C. (19, 97) D. (19, – 97) E. (1997, 0)

3. The number of geese in a flock increases so that the difference between the population in year $n+2$ and year n is directly proportional to the population in year $n+1$. If the populations in the years 1994, 1995, and 1997 were 39, 60, and 123, respectively, then the population in 1996 was

   A. 81 B. 84 C. 87 D. 90 E. 102

4. Medians BD and CE of triangle ABC are perpendicular. BD = 8 and CE = 12. What is the area of triangle ABC?

   A. 24 B. 32 C. 48 D. 64 E. 96

5. A line $x = k$ intersects the graphs of $y = log_5 x$ and of $y = log_5 (x+4)$. The distance between the points of intersection is 0.5 . Given that $k = a + \sqrt{b}$ where a and b are integers, what is $a + b$?

   A. 6 B. 7 C. 8 D. 9 E. 10

6. 30-60-90 right triangle OBC has O at the origin with B and C on the positive x and y axes, respectively. OB = 1. A circle with center P(h, k) with h > 1 is tangent to the coordinate axes and to AC. What is the radius of the circle?

   [Any exact answer please, sorry no choices ☹]

7. Let ABCD be a parallelogram and let AA', BB', CC', and DD' be parallel segments in space on the same side of the plane determined by ABCD with AA' = 10, BB' = 8, CC' = 18, and DD' = 22. If M and N are midpoints of A'C' and B'D', then MN equals

   A. 0 B. 1 C. 2 D. 3 E. 4

8. Let N be the least positive multiple of 20 such that N^2 is a perfect cube and N^3 is a perfect square. What is the number of digits of N?

   A. 3 B. 4 C. 5 D. 6 E. 7

9. A palindrome between 1000 and 10,000 is chosen at random. What is the probability that it is divisible by 7?

   A. 1/10 B. 1/9 C. 1/7 D. 1/6 E. 1/5

10. For what value of x does: $\log_{\sqrt{2}} \sqrt{x} + \log_2 x + \log_4 x^2 + \log_8 x^3 + \log_{16} x^4 = 40$?

    A. 8 B. 16 C. 32 D. 256 E. 1024

11. In ΔABC, $\cos(2A - B) + \sin(A + B) = 2$ and $AB = 4$. What is BC?

A. $\sqrt{2}$ B. $\sqrt{3}$ C. 2 D. $2\sqrt{2}$ E. $2\sqrt{3}$

12. For how may ordered triples (x, y, z) of nonnegative integers less than 20 are there exactly two distinct elements in the set $\{ i^x, (1+i)^y, z \}$, where $i = \sqrt{-1}$?

A. 149 B. 205 C. 215 D. 225 E. 235

13. Positive integers a, b, and c are randomly and independently selected from the set {1, 2, 3, …,2010}. What is the probability that $abc + ab + a$ is divisible by 3?

A. 1/3 B. 19/81 C. 31/81 D. 11/27 E. 13/27

14. The entries in a 3x3 array include all the digits from 1 through 9, arranged so that the entries in every row and every column are in increasing order. How many such arrays are there?

A. 18 B. 24 C. 36 D. 42 E. 60

15. A basketball game between the Wolverines and Tigers was tied at the end of the 1st quarter. The number of points scored by the Wolverines in each of the 4 quarters formed an increasing geometric sequence. The number of points scored by the Tigers in each of the 4 quarters formed an increasing arithmetic sequence. At the end of the 4th quarter, the Wolverines won by one point. Neither team scored more than 100 points. What was the total number of points scored by the two teams in the first half?

A. 30 B. 31 C. 32 D. 33 E. 34

16. A geometric sequence $\{ a_n \}$ has $a_1 = \sin(x)$, $a_2 = \cos(x)$, and $a_3 = \tan(x)$ for some real number x. For what value of n does $a_n = 1 + \cos(x)$?

A. 4 B. 5 C. 6 D. 7 E. 8

ANSWERS AND SOME *HINTS* – AMC 12 SAMPLE

1. D 29 Show that she scored at most 84 points in her first 5 games.

2. E (1997, 0) The slope and the y-intercept must be both positive or both negative. Which half-axis can such a line never intersect?

3. B 84 Let x be the number of geese in 1996 and let k be the *constant of proportionality*. Later, to factor $x^2 - 39x - 3780$, note that 3780 is a bit larger than 60^2 and that the two factors must differ by about 40. So the absolute value of the two factors are slightly larger than 40 and 80.

4. D 64 Remember that the intersection of the medians divides each median into the ratio 2:1. Thus EG = 4 and GC = 8. Also note that triangles DBA and DBC have the same area.

5. A 6 $y_2 - y_1 = \log_5 (x+4) - \log_5 x = 1/2$, then solve for x.

6. $\frac{3+\sqrt{3}}{2}$ or $1+\sqrt{1+\frac{\sqrt{3}}{2}}$ or equivalent form. Let D and E be the points of tangency on the y and x axes and let F be the point of tangency on BC. Tangents from a point to a circle are equal in length; hence CD = CF = y ; BE = BF = x ; x + y = 2 ; and $r = x + 1 = y + \sqrt{3}$. Cleverly, use $r = \frac{x + 1 + y + \sqrt{3}}{2}$

7. B 1 Though not required, it is easiest to visualize the solution if you assume that AA' and it's parallels are perpendicular to the plane. Note that ACC'A' and BDD'B' are trapezoids. Let O be the intersection of AC and BD. Therefore, OM and ON are the midlines of the two trapezoids. Also, show that O, M, and N are collinear.

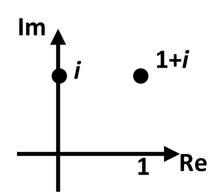
8. E 7 Let $N = 2^2 * 5 * 2^x * 5^y = 2^{2+x} 5^{y+1}$ To be a cube, x = 1, 4, 7… and y = 2, 5, 8… Analyze N^3 similarly. Together they yield x = 4 and y = 5.

9. E 1/5 $\underline{abba} = 1001a + 101b$. Since 1001 is a multiple of 7, *a* can be 1 through 9. Since 101 is not a multiple of 7, *b* must be a multiple of 7; namely 0 or 7.

10. D 256 Using $\log_b a = \frac{\log_n a}{\log_n b}$, convert all logs to base 2.

11. C 2 Since the maximum value of both sin x and cos y is 1, $\cos(2A - B) + \sin(A + B) = 2$, forces $\cos(2A - B) = 1$ and $\sin(A + B) = 1$ or 2A – B = 0 and A + B = 90. Solving, ABC is a 30-60-90 triangle.

12. D 225 Since $|i| = 1$ and $|1+i| = \sqrt{2}$; $i^x = (1+i)^y$ only when $i^x = 1$ and $(1+i)^y = 1$. Thus y=0, x = 0,4,8,12,16, and z is 0 or 2 through 19; for <u>95 solutions.</u> $i^x = z$ also only when $i^x = 1$ and $z = 1$. Same 5 solutions for x while y can be 1 through 19. <u>95 more solutions</u>. $(1+i)^y$ is real whenever *y* is a multiple of 8: 0, 8, 16 … Only $y = 0$ or $y = 8$ yield a z less than 20, namely $(1+i)^0 = 1$ and $(1+i)^8 = 16$. If y=0, z=1, then x is not a multiple of 4 for 15 solutions. If y=8, z = 16, x is any number 1 through 20. 15+20 = <u>35 solutions.</u>



13. E 13/27 $abc+ab+b = a[b(c+1)+1]$. If *a* is a multiple of 3, then done. If *a* is not a multiple of 3 then $b(c+1)+1$ must be a multiple of 3. Consider the cases when b = 0, 1, or 2 mod 3. $\frac{1}{3}+\frac{2}{3}[\frac{1}{3}*\frac{1}{3}+\frac{1}{3}*\frac{1}{3}]=\frac{13}{27}$

14. D 42 The digits 1, 2, 3, 4 must be arranged either in a 2x2 square in upper left corner [which can occur in 2 ways] or in an L-shaped piece along left side or top [each can occur in 3 ways]. Likewise, 6, 7, 8, 9 must be arranged in a 2x2 square in lower right corner [two ways] or in an L-shaped piece along bottom or right side [each can occur in 3 ways]. "5" always works in the 9th cell. Each of the two 2x2 pieces for 1-4 in upper left can be paired with any of the 6 L-shaped pieces for 6-9. OR Each of the six L-shaped pieces for 1-4 can be paired with two 2x2 pieces or three L-shaped pieces for 6-9. 2*6 + 6*5 = 42.

15. E 34 Since each team scored the same number of points in the first quarter, assign: Wolverines: a, ar, ar^2, and ar^3 in each quarter. Tigers: $a, a+d, a+2d,$ and $a+3d$ in each quarter. Therefore, $a(1+ r + ar + ar^2) = 4a + 6d + 1$. This implies that r is a positive integer and for each score to be less than 100, try r=2. Solving, a=5 and d=9. They scored 5-5 in first quarter and 10-14 in 2nd quarter for 34 points total.

16. E 8 From the first three terms: the common ratio $r = \cos(x)/\sin(x) = \tan(x)/\cos(x)$. With this common ratio, we can write the sequence as a ratio of sines and cosines. Also, the second equality yields $\cos^3 x = \sin^2 x$. We now need to write the 'goal' $\boldsymbol{\cos(x) + 1}$ as such a ratio. $\cos^3 x = \sin^2 x = 1 - \cos^2 x$ or $\cos^3 x + \cos^2 x = 1$. So $\cos^2 x\,[\cos x + 1] = 1$ and $\boldsymbol{\cos(x) + 1 = 1/\cos^2 x}$. As you write each term as the ratio of sines and cosines, using $\cos^3 x = \sin^2 x$ to simplify. Then the **8th** term can be written $1/\cos^2 x$.