Math Circle - March 24, 2019

An Introduction to Knot and Link Theory

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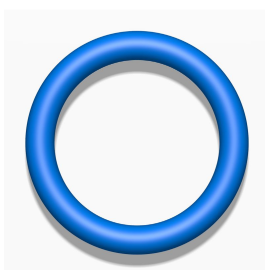
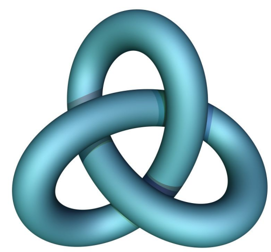
What is a knot? What is a link?

“A **knot** is just such a knotted loop of string, except that we think of the string as having no thickness, its cross-section being a single point. The knot is then a closed curve in space that does not intersect itself anywhere.” **(Adams, 1994, p. 2)**

That is, a knot is an embedding of a circle S^1 in 3-dimensional Euclidean space.

Examples:

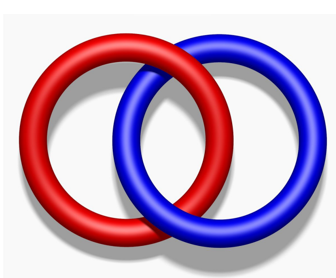
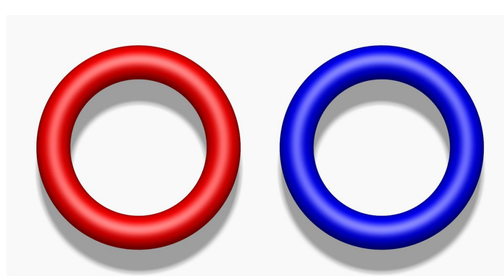




Unknot (U) **Trefoil (T)** **Trefoil* (T*)**

On the other hand, a **link** is a collection of disjoint knots, but which may be linked up, or knotted up, together.

Examples:

Hopf Link (H) **2-component Unlink (2-U)**

Problem 1: Together with your assigned group, and using the given materials, construct the previous given projections of U, T, T* and H.

A very brief summary of the history of knot theory

1880s:

- It was believed that a substance called ether filled all of space.

- Lord Kelvin (William Thomson, 1824-1907) claimed that atoms = knots (made of ether).

- This motivated Scottish physicist Peter G. Tait (1831-1901) to try to list all of the possible knots, which would result in a table of the elements.

- But in 1887, Michelson-Morley experiment proved that Kelvin was wrong.

- Chemists lost interest in knots for the next 100 years (because of Bohr's model).

1980s:

- Biochemists discovered knotting in DNA molecules.

- Synthetic chemists claimed that could be possible to create knotted molecules, where the type of knot determines the properties of the molecule.

In the present, knot theory have several applications to chemistry and biology.

In the meantime, after chemists lost interest in knots (1880s), mathematicians became intrigued with knots. Two main questions are the following:

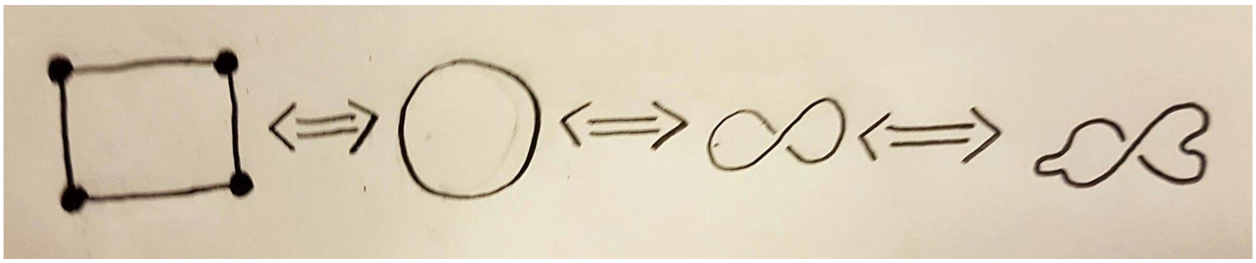
- How to classify knots?
- How to tell if two knots are isotopic (i.e., the same)?

Mathematicians make a lot of progress with something called **knot invariants**. We'll learn a little bit of this Today.

Ambient Isotopy

“We will not distinguish between the original closed knotted curve and the deformations of that curve through space that do not allow the curve to pass through itself. All of these deformed curves will be considered to be the same knot. We think of the knot as if it were made of easily deformable rubber.” **(Adams, 1994, p. 2)**

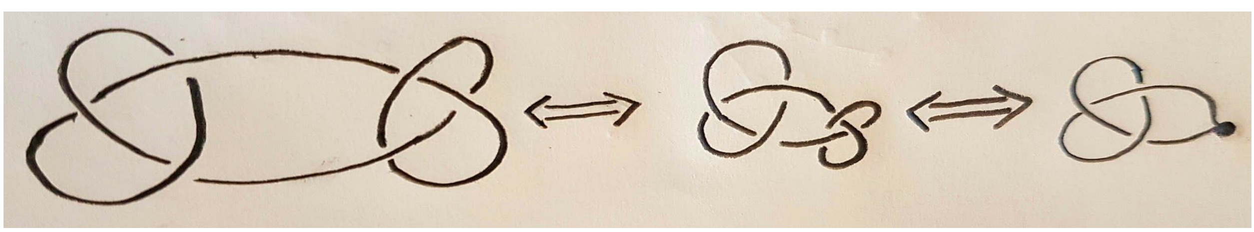
Example:



These are examples of **ambient isotopy**.

Definition: The movement of the string through three-dimensional space without letting it pass through itself is called an **ambient isotopy.**

Problem 2: Do all the following pictures describe a sequence of ambient isotopies? Justify your answer.

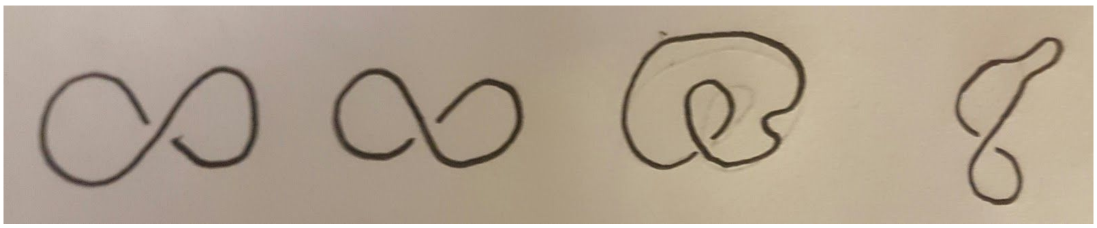


Answer:

Definition: “The places where the knot crosses itself in the picture are called the **crossings** of the projection.” **(Adams, 1994, p.3)**

Definition: “A deformation of a knot projection is called a **planar isotopy** if it deforms the projection plane as if it were made of rubber with the projection drawn upon it.” **(Adams, 1994, p. 12)**

Note that all the following are one-crossing projections of U, since we only need to untwist the single crossing (and apply planar isotopy):



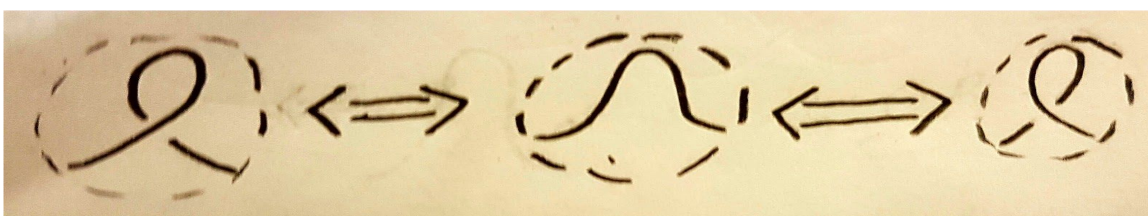
One-crossing projections of U

Reidemeister Moves

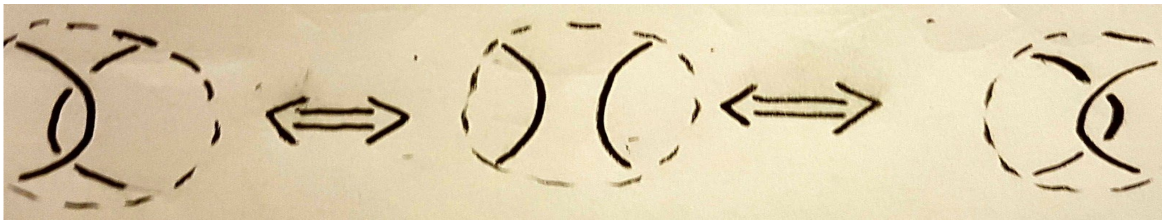
Definition: “A **Reidemeister move** is one of three ways to change a projection of the knot that *will* change the relation between the crossings.” **(Adams, 1994, p.13)**

Note: We will assume that the given projection of the knot only changes in the enclosed area depicted in the figure.

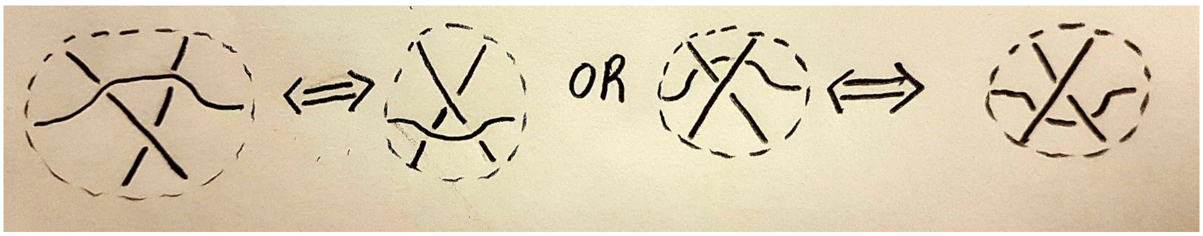
Reidemeister I **(R-I)**: twist or untwist



Reidemeister II **(R-II)**: add two crossings or remove two crossings



Reidemeister III **(R-III)**: slide a strand from one side of a crossing to the other side of the crossing



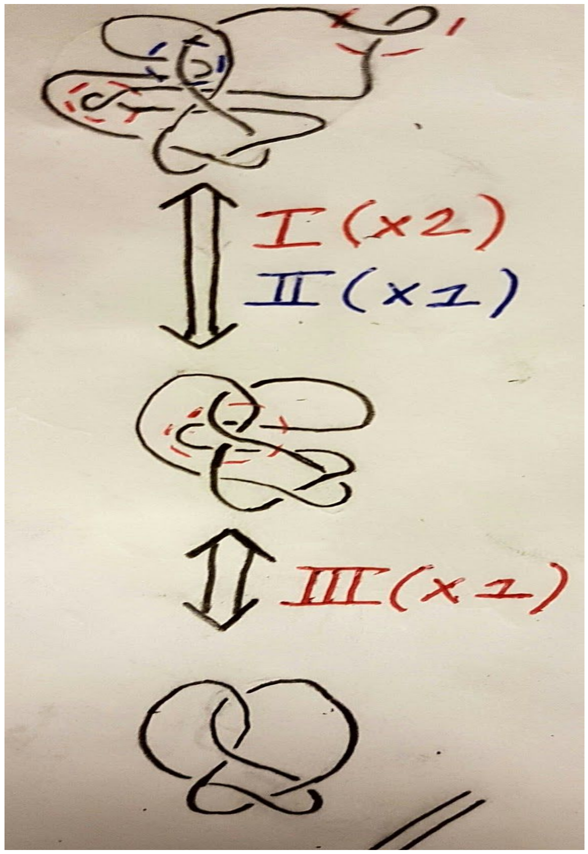
In 1926, the german mathematician Kurt Reidemeister (1893-1971) proved the following fact:

Theorem: If we have two distinct projections of the same knot, then we can get from the one projection to the other by a series of Reidemeister moves and planar isotopies.

Example:

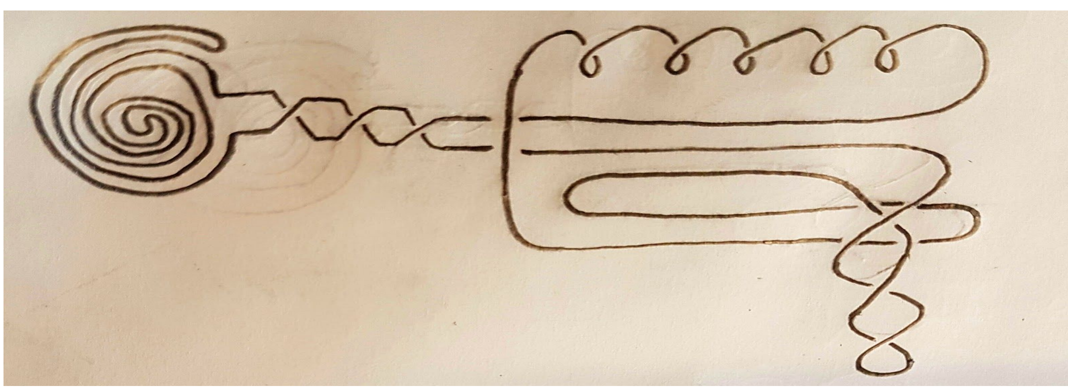
Show that the two projections in the following figure represent the same knot by finding a series of Reidemeister moves from one to the other.

A solution: (Recall that knots are "unaffected" by ambient isotopies. In particular, by planar isotopies.)



Problem 3:

Show that the projection in the following figure represent the unknot U by finding a series of Reidemeister moves (and planar isotopies). **You must use all three Reidemeister moves.**



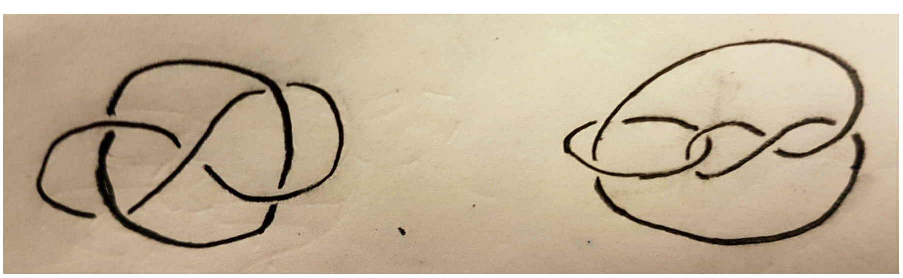
Answer:

Equality of links

"Two links are considered to be the same if we can deform the one link to the other link without every having any one of the loops intersect itself or any of the other loops in the process."

(Adams, 1994, p. 17)

Problem 4: Explain why the two following projections represent the same link, which is called Whitehead link.



Answer:

Invariants

Definition: A **knot (or link) invariant** is a quantity (in a broad sense) defined for each knot (or link) which is unchanged by ambient isotopy. That is, equivalent knots (or links) have the same invariant.

Remarks:

- If two knots (or links) have the same invariant that doesn't mean that they are the same knot (or link). Today we'll see an example of this.
- There are many discovered invariants, but Today we'll study three of them: **crossing number**, **linking number** and the **Alexander-Conway polynomial**.

Crossing Number

Definition: "The crossing number of a knot K, denoted c(K), is the least number of crossings that occur in any projection of the knot." **(Adams, 1994, p. 67)**

Problem 5: Why is c(K) a knot invariant?

Answer:

Problem 6: Explain why there are no one-crossing, nor two-crossing, nontrivial knots.

Answer:

Determining the crossing number of a knot K

For this, we start by considering a projection of the knot K with some number of crossings n. Then, by definition, $c(K) \leq n$. Now, if all of the knots with fewer crossings than n are known, and if K does not appear in the list of knots of fewer than n crossings, then $c(K)=n$. Today we will learn a little bit about Knot Tabulation, but for more you can take a look at Chapter 2 of Adams (1994). //

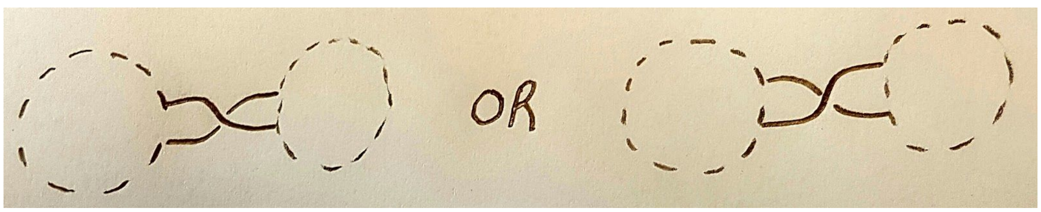
Reduced alternating projection of a knot K

Definition: “An **alternating knot** is a knot with a projection that has crossings that alternate between over and under as one travels around the knot in a fixed direction.” **(Adams, 1994, p. 7)**

Examples: T and T* (look at page 1)

Definition: “Call a projection of a knot **reduced** if there are no easily removed crossings, […]” **(Adams, 1994, p. 68)**

That is, we don’t find a crossing that looks like the following:



In 1986, Lou Kauffman (University of Illinois at Chicago), Kunio Murasugi (University of Toronto) and Morwen Thistlethwaite (University of Tennessee) proved the following fact:

Theorem: A reduced alternating projection of a Knot K exhibits c(K) crossings.

Problem 7: For each one of the previous projections of U, T and T* (pages 1 and 6) determine if the given projection is a reduced alternating projection. If no, explain why. If yes, determine c(K). Do the same with the following projections of 8 and 3T. Compare each possible pair of the previous five knots: (U,T), (U,T*), (U,8), (U,3T), (T,T*), (T,8), (T,3T), (T*,8), (T*,3T), (8,3T). What can you conclude in each case?

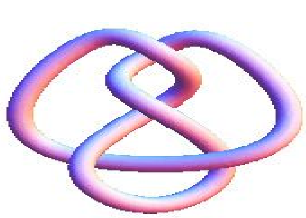
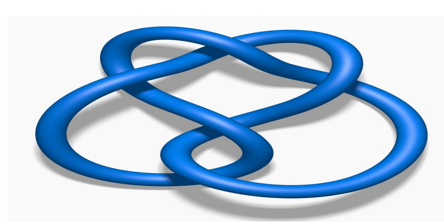


Figure eight (8)



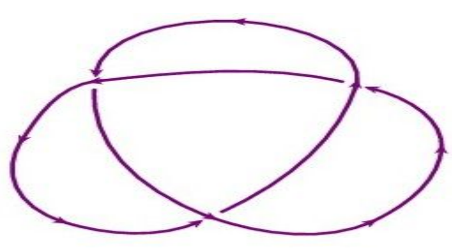
Three-twist (3T)

Answer:

Linking Number

Definition: "An **orientation** is defined by choosing a direction to travel around the knot. This direction is denoted by placing coherently directed arrows along the projection of the knot in the direction of our choice. We then say that the knot is **oriented**." **(Adams, 1994, p.10)**

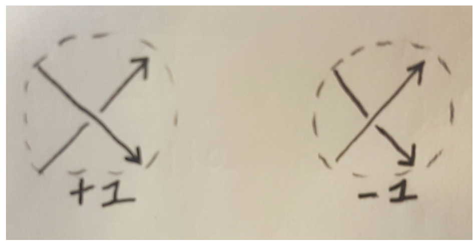
Example:



Oriented T*

Definition: Let K_1 and K_2 be two components in a link. Their **linking number** is defined as follows:

- Choose an orientation for each of K_1 and K_2.
- At each crossing of K_1 with K_2, assign +1 or -1 according to the following figure:



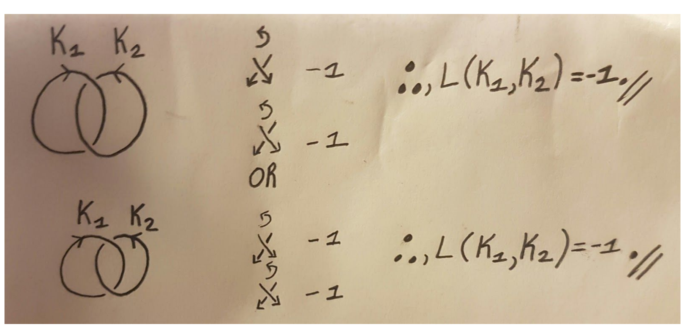
- Add up all the previous +1 and -1, and divide by 2.

Define the **linking number of** K_1 **and** K_2, denoted by L(K_1,K_2), to be the previous number.

Remark: "Note that if a crossing is of the first type, then rotating the understrand clockwise lines it up with the overstrand so that their arrows match. Similarly, if a crossing is of the second type, then rotating the understrand counterclockwise lines the understrand up with the overstrand so that their arrows match." **(Adams, 1994, p.19)**

Example: Compute the linking number of the two components of H for two possible orientations.

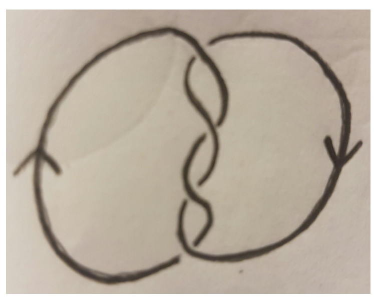
A solution:



Problem 8: Compute the linking number of the two components of H for the other two possible orientations. What can you conclude about L(K_1, K_2)?

Answer:

Problem 9: Compute the linking number of the two components in the following oriented link L:

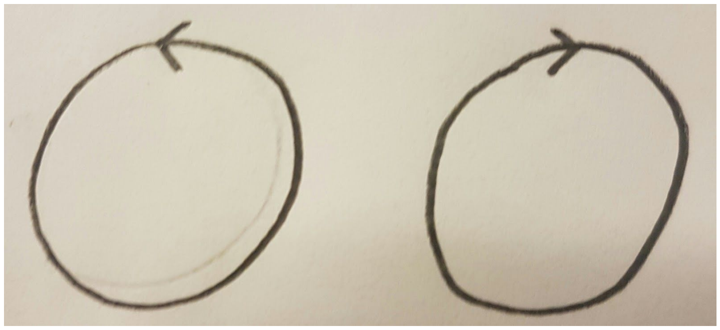
.

Answer:

Problem 10: In this problem, we'll conclude that the linking number is an invariant of the oriented link**.** That is, once the orientations are chosen on the two components of the link, the linking number doesn't change by ambient isotopies. After studying page's 5 theorem, it is not difficult to understand that we can get from any one projection of a link to any other by a series of Reidemeister moves. Explain why two different projections of the same oriented link yield the same linking number. (**Hint:** Note that applying any of R-I, R-II or R-III to a single component of the link doesn't affect the linking number, so you only care about Reidemeister moves applied to strands of the different components.)

Answer:

Problem 11: Compute the linking number of the two components in the following oriented projection of 2-U. Compare, via $|L(K_1, K_2)|$, H, L and 2-U. What can you conclude?



Answer:

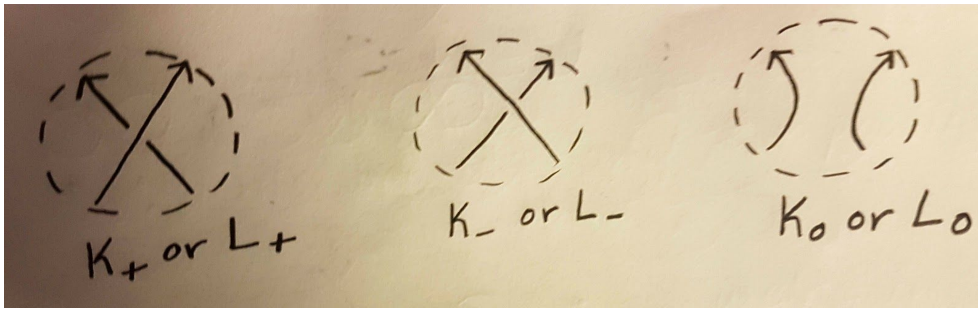
Alexander (1920's)-Conway Polynomial (1970's), ∇

∇ is a polynomial for oriented knots and links, and we describe it so that its variable is t.

Definition: For an oriented knot (or link), we have the following rules:

- Rule 1: ∇(U)=1, where we consider *any* projection of U

- Consider three projections of knots (or links) K_+, K_- and K_0 (or L_+, L_- and L_0) that are identical, except in the region depicted in next figure:



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Rule 2: $\nabla(K_+)$ - $\nabla(K_-) = x\nabla(K_0)$.

After see this definition, the "most natural" question could be the following:

- Why is this an invariant for oriented knots (or links)? That is, why the calculation of the polynomial cannot depend on the particular projection that we start with? For this, we know that *it must be unchanged by the Reidemeister moves*.

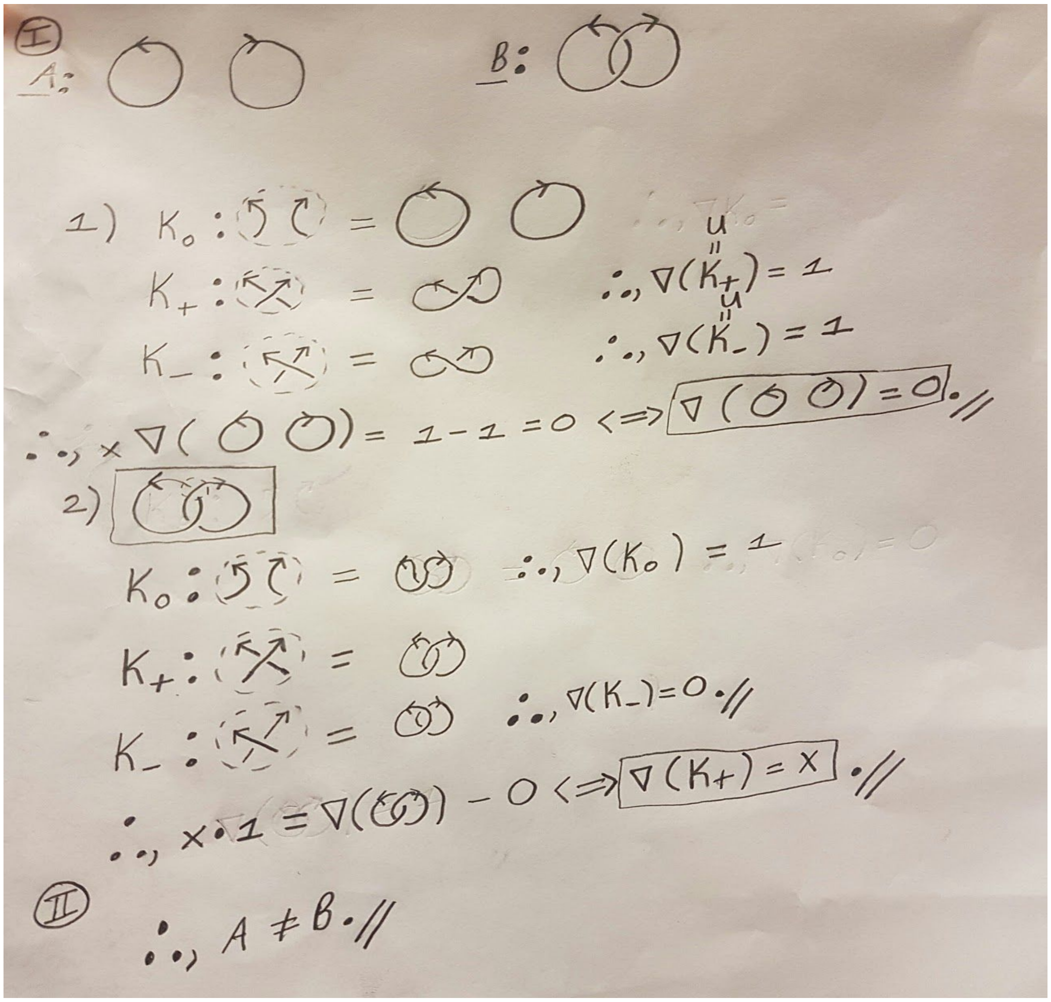
Today we're not going to show that this is a knot (link) invariant. But let's think about with examples/problems.

Example: Determine ∇ for the following two two-component oriented links:


.

What can you conclude about them?

Solution:



Problem 12: Reverse the orientation of one of the circles in previous oriented H and compute ∇.

Answer:

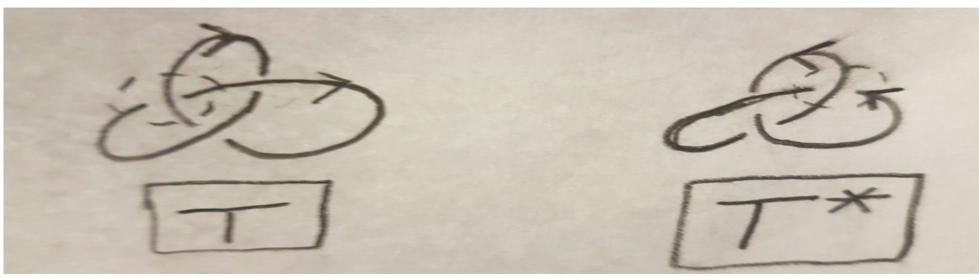
Problem 13: For the given example of the oriented Hopf link (before problem 12), choose the other crossing and compute ∇.

Answer:

Problem 14: Show that ∇(n-U)=0, for any $n \geq 3$ (n is a natural number).

Answer:

Problem 15: Compute ∇ for the following oriented projections of T and T* using the specified crossing. What can you conclude? Now, change the orientation of T* and compute ∇. Then, compare with your previous computation of T. What can you conclude?



Answer:

Fact: T and T* are not ambient isotopic!!!!!!!!!!

Extra problems (Homework):

(Challenge) Problem 16: Compute $\nabla(8)$.

(Challenge) Problem 17: Prove previous fact.

Sources used/Acknowledgment:

1) Dr. Renato Feres (Wash. U. in St. Louis), Dr. Dave Auckly (Kansas State University) and Dr. Blake Thornton (Wash. U. in St. Louis) gave me advising in how to prepare a presentation for a Math Circle. Thanks to Dr. Rachel Roberts (Wash. U. in St. Louis) that suggested me the advising of two of the previous professors to prepare for this Math Circle activity.
2) Most of what I have written here, I learned it from two sources: *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots* (Dr. Colin C. Adams, 1994) and the lecture *Knots and Surfaces I* (Dr. Norman J. Wildberger, 2011) posted on YouTube.
3) The pictures used in problems 1) and 2) comes from Wikipedia, WordPress and WolframMathWorld. The picture of **oriented T*** comes from American Mathematical Society (ams.org).