Seifert surfaces and genera of knots

Michael Landry

These are the lecture notes for a colloquium talk on knot theory given to undergraduates at the Summer Undergraduate Mathematics Research at Yale (SUMRY) program. Since I learned knot theory from Colin Adams' excellent book [1], the treatment here is similar, but with my own examples and spin.

1 Knots

Whatever the definition of a knot is, it should rigorously model the following situation: you have a string which is tangled up in some way, and you attach the 2 loose ends to each other. This could be by gluing them, taping them, or by tying a knot, although the last method is linguistically problematic for obvious reasons. The result is a circular string which you can play with, and which we will call a *knot*. Maybe if you fiddle around with it for long enough, you can untangle it to a circle which lies flat on the table—no cutting allowed though—in this case, the knot you created is what we call the *unknot*. Otherwise, your string is knotted in some intrinsic sense. This knotting phenomenon is what knot theory explores.
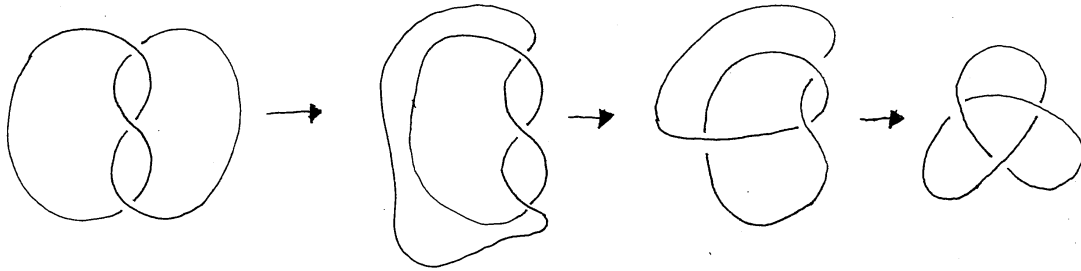

Figure 1: These are all the same knot, as the deformations above show.

If you do not like thinking of your mathematical objects being made out of string and tape, fine. Being more explicit, but still a little vague, think of a knot as an

equivalence classes of embeddings of the circle in three-dimensional space, where 2 embeddings are equivalent if one can be continuously deformed to the other such that each intermediate map is injective.

2   Connected sums of knots

We now define an operation Knots $\times$ Knots $\rightarrow$ Knots called *connected sum*. The connected sum of 2 knots, K_1 and K_2, denoted $K_1 \# K_2$, is obtained in the following way: draw the two knots next to each other, and remove a small segment from an outer arc of each knot. Now glue the 2 remaining parts together as in Figure 2. It
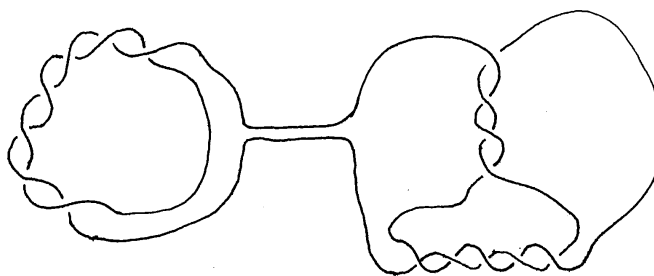

Figure 2: A connected sum of 2 knots.

is not clear that this operation is well-defined, and in fact it is not unless we define it on *oriented* knots, where the gluing is determined by requiring that it preserve orientation. You might be a little nervous that we had to make a choice of which small segment to remove. This is not an issue as long as the segments we choose do not traverse any crossings in our drawings of the knots. Can you prove this?

What can we say about $\#$? Here are some observations:

(a) $\#$ is commutative.

(b) $\#$ is associative.

(c) if K is any knot and U is the unknot, then $K \# U = K$.

One next natural question to ask is whether $\#$ has inverses. That is, does there exist a knot K^{-1} such that $K \# K^{-1} = U$? Could a knot like the one shown in Figure 2 be the unknot? It turns out that the answer is no, and the rest of this talk will be devoted to proving this result.

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3 The classification of compact, connected, orientable surfaces

We need to make a few definitions.

A *surface* is a two-dimensional manifold. If you are not comfortable with that description, think of a surface as a space that looks like a two-dimensional plane if you zoom in really close to any point. A *surface with boundary* is a space that looks like either a two-dimensional plane or the upper half of a two-dimensional plane if you zoom in on any point. The *boundary* of such a surface is composed of the "upper half plane" points. A surface is *connected* if any 2 points in the surface can be connected by a path which is also in the surface. A surface is *compact* if every sequence of points in the surface has a convergent subsequence.

Define an *orientable surface* to be a surface (possibly with boundary) which is 2-sided, in the sense that if a tiny man is standing on the surface at a point P and takes a walk which stays away from the boundary, it is impossible for him to return to P with his head pointing in the opposite direction.

Remark 1. Orientability is usually defined in a different, more complicated way. But for surfaces embedded in an orientable 3-manifold, orientability and 2-sidedness are equivalent conditions. However, there are examples of three-dimensional manifolds which are nonorientable and admit two-sided embeddings of nonorientable surfaces (see [5], for example). Since we are picturing our surfaces as sitting in $\mathbb{R}^3$, which is orientable (note that I have not defined orientability for three-dimensional manifolds!), we are good to go.

All those definitions pay off in the following classification theorem.

Theorem 2. *Every orientable, connected, compact surface with boundary is topologically equivalent (or homeomorphic, if you know that word) to one of the surfaces shown in Figure 3.*

That is, any such surface is uniquely determined by its number of boundary components and its number of "holes," which we call the *genus* of the surface (the plural of genus is *genera*). For a proof of Theorem 2, see [4].

4 Seifert surfaces

A Seifert surface for a knot K is a compact, orientable, connected surface with boundary equal to K. By the previous section, any such surface must be topologically
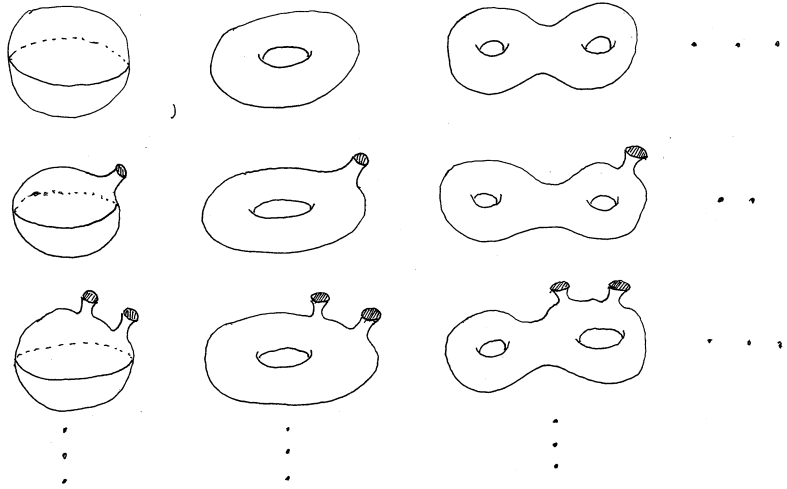
Figure 3: Classification of orientable, connected, compact surfaces with boundary.

equivalent to one of the surfaces in the second row of Figure 3 since it has only one boundary component. Therefore any Seifert surface has a well-defined genus. The genus of a knot K, denoted $g(K)$, is the minimum genus of all Seifert surfaces for K.

Note that the definition above does not make sense unless we know that any knot possesses at least one Seifert surface. But this fact is not that hard to prove. In fact, Seifert himself gave a nice algorithm for producing a Seifert surface given a drawing of a knot, which is illustrated in Figure 4.

1. Pick an orientation for your knot.

2. Start tracing around the knot in your chosen direction. When you get to a crossing turn right or left, in whichever direction agrees with the orientation of the knot. When you get back to where you started, start somewhere else in the drawing. Do this until you've traced over all the arcs in the drawing.

3. In the previous step you created a collection of circles, called Seifert circles. Attach a disk to each Seifert circle, so that each Seifert circle now bounds a disk. In the case of concentric circles, vary the height of the disks so that the innermost Seifert circles are the highest.

4. At the site of each of the old crossings, attach the 2 neighboring disks by gluing in a rectangular band with a half twist, the direction of the twist being determined by the original crossing.
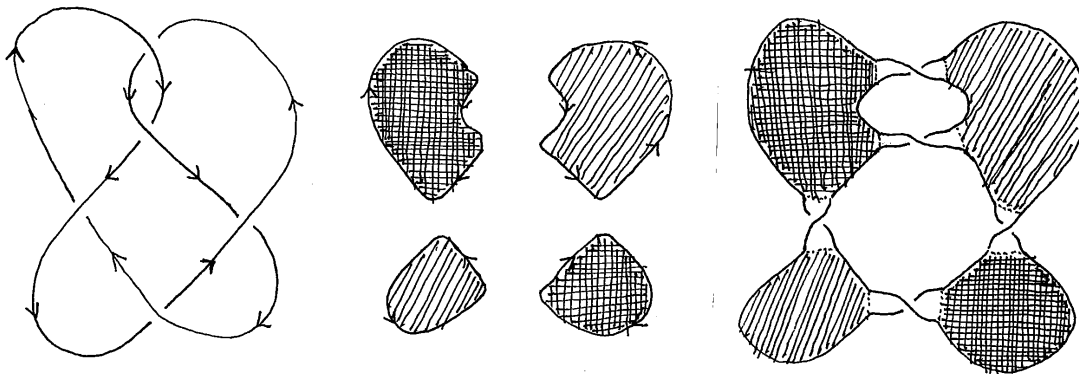
Figure 4

The resulting surface clearly has boundary the original knot. What is not imme-
diately clear is that the resulting surface is orientable. But this is easy enough to
see, and I leave it as an exercise for you, dear reader. Here's a hint: if the boundary
of a disk has counterclockwise orientation, paint its top red and its bottom blue. Do
the opposite for disks whose boundaries have clockwise orientation. Why does the
way we glued in the bands allows us to extend the red and blue paint over the entire
surface such that the two colors never meet?

Example 3. Consider the Seifert surface for the trefoil knot shown Figure 5. By
cutting and pasting as shown in the figure, we can see that this surface is homeo-
morphic to torus minus a disk. You can also see this using Euler characteristic, but
cutting and pasting feels more fun to me.

This shows that the genus of the trefoil knot is ≤ 1. If the trefoil knot had genus
0, then it would bound a disk, but this is impossible (why?). Therefore we have
determined that the genus of the trefoil knot is 1.

Example 4. Let's kick it up a notch. Consider the Seifert surface shown in Figure 6.
Well, according to our definition this is not strictly speaking a Seifert surface because
its boundary is not a knot, it is a *link*. A link is just a collection of circles embedded
in 3 space, up to the same equivalence relation that we used to define knots. We
broaden our definition of Seifert surface to include those whose boundaries are links.
This particular link is called the Borromean rings, and has the neat property that
no two of the three circles are linked to each other, even though the link as a whole
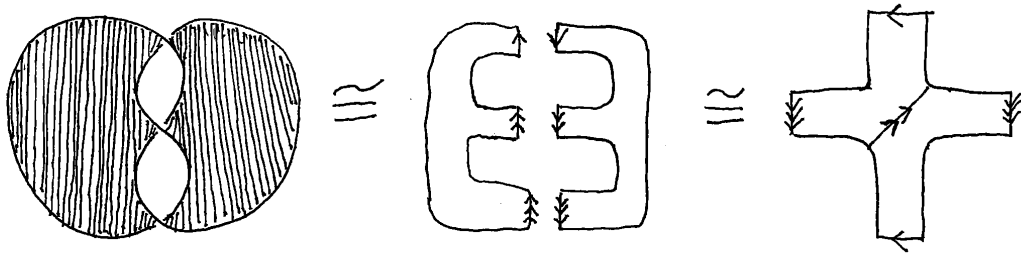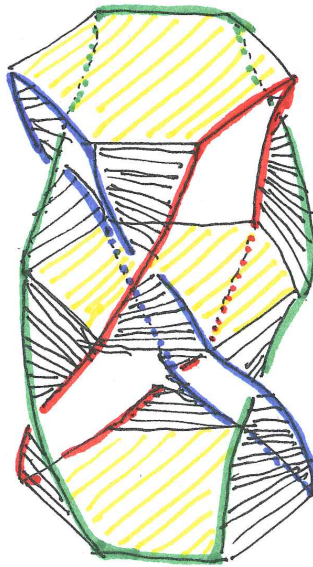cannot be undone.

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Figure 5

Figure 6: A Seifert surface for the Borromean rings.

We can obtain the surface in Figure 6 by gluing together 3 hexagons in the way prescribed by Figure 7, which shows that the surface is topologically equivalent to a torus minus 3 disks. Therefore the genus of the Borromean rings is ≤ 1.

Note that "topologically equivalent" does not mean that we can continuously and injectively deform the surface in Figure 6 (à la our knot deformations) to look like the $(4, 2)$ entry of the table in Figure 3. Indeed, such a deformation would unlink the Borromean rings, which is impossible. We will not prove that, although it is not hard—the 3 component unlink has a property called *tricolorability* which the
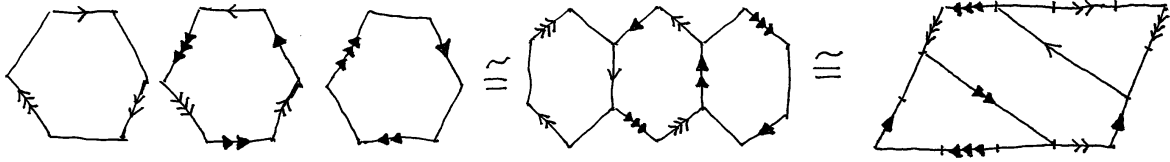
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Figure 7

Borromean rings do not have.

One last thing about this example—we have shown that the genus of the Borromean rings is at most one, but we have not shown that the Borromean rings do not bound a thrice-punctured sphere, so we cannot conclude that their genus is actually 1. The genus is in fact 1, but the only proof I am aware of uses the Thurston norm on homology, which falls beyond the scope of this talk.

5   Main Theorem

It is time to finally get around to resolving that question about "inverses" of knots from Section 2. Conveniently, this follows immediately from the following theorem describing the relationship between genus and connected sum.

Theorem 5. *Let K_1 and K_2 be knots. Then $g(K_1 \# K_2) = g(K_1) + g(K_2)$.*

Proof. First, observe that $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$ because you can attach minimal genus Seifert surfaces for K_1 and K_2 by a rectangular band to get a Seifert surface for $K_1 \# K_2$ whose genus is the sum of the original two.

Now we prove that the reverse inequality holds. Take a Seifert surface Σ for $K_1 \# K_2$ which is of minimal genus. By assumption, there exists a sphere S, punctured twice by $K_1 \# K_2$, which separates K_1 from K_2. By perturbing the sphere we preserve this separation property and guarantee that Σ intersects S in arcs, all but one of which are circles (the last one is a path P between the 2 punctures). The fact that we can do this is a property of *transverse* intersections, which you can read about in [3] or simply take on faith.

If there are no circles on the surface of S, we are done—by cutting along P we do not change the genus and we obtain Seifert surfaces for K_1 and K_2, so $g(K_1 \# K_2) \geq g(K_1) + g(K_2)$.

Otherwise there is an innermost circle on S, and by doubling S inside the circle we create a new Seifert surface which intersects S in one less circle. If the resulting

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surface is disconnected, discard the component which does not connect to the knot. This surgery does not increase the genus, because it increases Euler characteristic by 2 without changing the number of boundary components. As an aside, this even shows that we always create a sphere component by this surgery, since $g(\Sigma)$ is minimal.

By repeating this type of surgery, we eventually reduce the number of intersecting circles to 0, so we are finished. $\qquad\square$

Corollary 6. *If either K_1 or K_2 is nontrivial, then $K_1 \# K_2$ is not the unknot.*

Proof. As was remarked earlier, the only knot whose genus is 0 is the unknot. $\qquad\square$

References

[1] Adams, Colin. *The knot book.* American Mathematical Society, 2001.

[2] Candel, Alberto and Conlon, Lawrence. *Foliations II.* American Mathematical Society, 2003.

[3] J. Lee. *Introduction to smooth manifolds.* Second edition. Springer, 2013.

[4] Massey, William. *Algebraic topology: an introduction.* Harcourt, Brace & World, Inc. 1967.

[5] Weeks, Jeffrey. *The shape of space.* Chapman & Hall, 2001.