Mathleague Target Round (2002, 4th Qualifier, Round 4) – 10 minutes, NO Calculator

7. What is the sum of the positive factors of 1001?

8. So for this exercise we want to make unit fractions add up to 1. For example, let's start with $\frac{1}{2}$, and to it we add the next unit fraction, $\frac{1}{3}$, to get $\frac{5}{6}$. The next unit fraction in line would be $\frac{1}{4}$, but we can't add it because that would put us over 1. So we look a the next one, $\frac{1}{5}$. Still too large. Now we look at $\frac{1}{6}$. We can add that in and not be greater than 1. In fact, adding $\frac{1}{6}$ gives us exactly 1, so we stop and $\frac{1}{6}$ is the last unit fraction we have added. If we follow a similar procedure with $\frac{1}{3}$ as our starting fraction, what is the last fraction we add?

Mathleague Relay Round (2001, 3rd Qualifier, Round 1) – 3 minutes, NO Calculator

(1-1) How many two-digit positive integers have two different digits?

(1-2) Let T = TNYWR. A given regular polygon has exterior angles with measure less than $T°$ (an exterior angle is the supplement of an interior angle of a polygon). What is the smallest number of sides the polygon could have?

(1-3) Let T = TNYWR. Compute the positive solution x of the quadratic equation $x^2 - 11Tx = 80T^2$.

20 minutes
(calculator)

Team

GPML 10005 Large School

1. Stacy has d dollars. She enters a mall with 10 shops and a lottery stall. First she goes to the lottery and her money is doubled, then she goes into the first shop and spends 1024 dollars. After that she alternates playing the lottery and getting her money doubled (Stacy always wins) then going into a new shop and spending \$1024. When she comes out of the last shop she has no money left. What is the minimum possible value of d?

2. If a and b are real numbers such that $a + b = ab = 5$, and $a > b$, then what is a?

3. Start with an angle of $60°$ and bisect it, then bisect the lower $30°$ angle, then the upper $15°$ angle, and so on, always alternating between the upper and lower of the previous two angles constructed. This process approaches a limiting line that divides the original $60°$ angle into two angles. Find the measure (degrees) of the smaller angle.

4. For what single digit n does 91 divide the 9-digit number $12345n789$?

5. A dart is thrown at a square dartboard of side length 2 so that it hits completely randomly. What is the probability that it hits closer to the center than any corner, but within a distance 1 of a corner?

6. If x, y, and z are distinct positive integers such that $x^2 + y^2 = z^3$, what is the smallest possible value of $x + y + z$.

7. Evaluate $\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n}$, where $\cos\theta = \frac{1}{5}$.

8. If I pick two two-digit integers at random, with replacement, then what is the probability that their sum is a multiple of 3?

9. For what value of $x > 0$ does the fraction $\frac{25+16x^4}{x^2}$ have the least value?

10. Find all the roots of $(x^2 + 3x + 2)(x^2 - 7x + 12)(x^2 - 2x - 1) + 24 = 0$.

(calculator) 2001, 4th
40 minutes Sprint

1. Which of the following is the value of the expression

$$5^5(5^5 + 5^5 + 5^5 + 5^5 + 5^5)?$$

A) 5^{10} B) 5^{11} C) 5^{25}
D) 5^{125} E) None of the above

2. What is the negation of the statement "Some day, I will eat lobster"?

A) On no day will I ever eat lobster.
B) Some day, I will not eat lobster.
C) I will not eat lobster everyday.
D) Some day, I might eat lobster.
E) None of the above

3. Suppose the triangle in the coordinate plane with vertices at (3, 4), (-2, 5), and (-1, -3) is rotated 180° about the origin. Which of the following is one of the vertices of the resulting triangle?

A) (1, 3) B) (2, 5) C) (3, 4)
D) (5, -2) E) None of the above

4. On an 8 x 8 checkerboard, each square is covered with a red checker, covered with a black checker, or empty. The average number of red checkers on each column is 2.5. The average number of black checkers on each row is 2. How many empty squares are there?

A) 20 B) 24 C) 28
D) More information is needed.
E) None of the above

5. Suppose that a, b, and c are natural numbers such that the mean, median, and mode of the numbers 1, 3, 3, a, b, and c are all integers. If the product of the mean, median, and mode is 74, then what is $a + b + c$?

A) 6 B) 76 C) 437
D) More information is needed.
E) None of the above

6. Two regular polygons, one with n sides and one with $n + 1$ sides, are drawn. Each interior angle of one of the polygons is exactly 4° larger than each interior angle of the other polygon. What is the value of n?

A) 7 B) 8 C) 9
D) 10 E) None of the above

7. There exist three distinct digits A, B, and C, and two three-digit natural numbers M and N containing each of these digits, such that the product of M and N is 160,000. What is the sum of the digits A, B, and C?

A) 14 B) 15 C) 16
D) 17 E) None of the above

8. For what value of k does the system of equations

$$2x - y + z = 0$$
$$3x + 2y - 5z = 0$$
$$x + 3y + kz = 1$$

have no solution?

A) -12 B) -6 C) 3
D) 6 E) None of the above

9. A Martian drives along a busy highway on Earth and notices a sign that says "SPEED LIMIT 55 MPH." However, most of the traffic is flowing at 65 MPH. The Martian assumes this is normal (and legal) because he assumes the sign is written in base b, where b is an integer. What is b?

A) 8 B) 9 C) 11
D) 12 E) None of the above

10. A particle starts at the origin in the coordinate plane and moves up or to the right one unit. It repeats this process, moving up or to the right one unit, until it reaches a point of the form (x, y) such that $x + y = 5$. How many different paths could the particle have taken?

A) 10 B) 20 C) 25
D) 32 E) None of the above

11. In a given square, a circle is inscribed. An equilateral triangle is then inscribed in the circle. If k is the ratio of the square's area to the triangle's area, what is k^{-2}?

A) 8 / 27 B) 16 / 81
C) 27 / 256 D) 81 / 256
E) None of the above

12. Suppose that a, b, and c are integers such that $a^2 + b^2 = c^2$. Which of the following must be true?

I. At least one of a and b is even.
II. At least one of a and b is divisible by 3.
III. At least one of a, b, and c is divisible by 5.

A) I only
B) II only
C) I and II only
D) I, II, and III
E) None of the above

13. Let N be the product of all prime natural numbers less than 30. What is the remainder when N is divided by 9?

A) 0 B) 2 C) 3
D) 6 E) None of the above

14. Four points are drawn in the plane so that no three are collinear. Two distinct line segments are drawn so that each segment has two of these points as endpoints. What is the probability that the segments intersect in at least one point, assuming that the endpoints of each segment are chosen at random?

A) 2 / 15 B) 1 / 5 C) 4 / 15
D) More information is needed.
E) None of the above

15. How many distinct five-digit integers of the form ABABA (where A and B are digits, with A nonzero) are divisible by 11?

A) 5 B) 6 C) 7
D) 8 E) None of the above

16. Let $ABCD$ be a square. Suppose E and F are drawn on sides AB and CD, respectively, so that AE = 2EB and CF = 2FD. Now, let G and H be drawn on segment EF so that each of the line segments AG and CH is perpendicular to EF. What is the ratio of the distance GH to the distance EF?

A) 2 / 5 B) 1 / 2 C) 3 / 5
D) 2 / 3 E) None of the above

17. Let "log" represent the base-10 logarithm, and "ln" the natural, or base-e, logarithm. If $x > 10^{10}$, then which of the following is the smallest?

A) $\log x$ B) $\log(\ln x)$
C) $\ln(\log x)$ D) $\ln(\ln x)$
E) None of the above

18. How many nine-digit positive integers having nine different digits are divisible by 9?

A) 322,560 B) 362,880
C) 685,440 D) 725,760
E) None of the above

19. Suppose that $\{x_n\}$ is a sequence of numbers such that $x_1 = 1$, $x_2 = 0$, and

$$x_n + 2x_{n+1} + x_{n+2} = 1$$

for each natural number n. What is the value of x_{2001}?

A) -1000 B) -999 C) 999
D) 1000 E) None of the above

20. Let $ABCD$ be a regular tetrahedron of volume 1. Suppose E, F, and G are the midpoints of edges AB, AC, and AD, respectively, and H is the centroid of face BCD. Consider the polyhedron with vertices A, E, F, G, and H, which has six triangular faces. What is the volume of this polyhedron?

A) 1 / 6 B) 1 / 4 C) 1 / 3
D) 1 / 2 E) None of the above

21. For what nonzero value of k do the graphs of the equations $y = x + k$ and $y = (x^3 - 2x^2) / |x|$ intersect in exactly two points?

A) -25 / 4 B) -4 C) -9 / 4
D) -1 E) None of the above

22. In the coordinate plane, a right triangle ABC (with right angle at vertex C) is drawn. This triangle is drawn so that vertex A is the origin, B is on the positive x-axis, and C is on the curve $y = x^{1/2}$. If the area of the triangle is 7 / 2, then to the nearest unit, what is the length of the hypotenuse AB?

A) 4 B) 5 C) 6
D) 7 E) None of the above

23. Suppose N is a six-digit natural number whose leftmost three digits are 1, 2, and 3 (in order from the left). When N is divided by any of the numbers 7, 8, and 9, it leaves a remainder of 6. If the hundreds digit of N is even, then what is the tens digit?

A) 3 B) 5 C) 6
D) 8 E) None of the above

24. Suppose that ABC is a triangle with an inscribed circle that touches the triangle at the points D, E, and F. If AB = 2, BC = 3, and CA = 4, then what is the ratio of the area of triangle DEF to that of triangle ABC?

A) 5 / 32 B) 3 / 16 C) 7 / 32
D) 1 / 4 E) None of the above

25. What is the sum of the positive proper divisors of 2751?

A) 1351 B) 1577 C) 1605
D) 1793 E) None of the above

26. Carrie writes down a natural number consisting of a one, followed by k zeroes, followed by another one. She observes that this number is divisible by 121. What is the smallest possible value of k?

A) 9 B) 13 C) 15
D) 19 E) None of the above

27. Seven coins are flipped, then arranged randomly in a circular pattern. What is the probability that there exist five consecutive coins on the circle which show heads?

A) 3 / 32 B) 13 / 128
C) 7 / 64 D) 15 / 128
E) None of the above

28. Suppose ABC is a triangle such that the measure of angle BCA is 45°, $AC^2 = 1$, and $AB^2 = 5 / 9$. If BC is of the greatest possible length (under these conditions), what is the area of triangle ABC?

A) 1 / 3 B) 2 / 5 C) 1 / 2
D) 3 / 5 E) None of the above

29. We will say a terminating decimal is *good* if it is between 0 and 1, and for some n, it has exactly n consecutive zeroes to the right of the decimal point, followed by exactly n consecutive digits which are either 3 or 6 (combinations of the two are allowed), followed by zeroes. (One such number is 0.000363.) Let the sum of all the *good* numbers be a / b, where a / b is in simplest form. What is $a + b$?

A) 437 B) 452 C) 477
D) 491 E) None of the above

30. Consider the triangle in the coordinate plane with vertices at (1, 1), (-1, -1), and (-2, -1). Suppose this triangle is rotated 360° about the x-axis. The set of all points the triangle passes through during this rotation forms a solid S. What is the volume of S?

A) 59π / 54 B) 61π / 54
C) 65π / 54 D) 67π / 54
E) None of the above

2019-2020 MCTM/Mathleague High School Contest Events

TEAM TEST: This is a ten question (five question for Division B schools), twenty minute test which a team of up to six works on together (or three for Division B schools). Each question will be worth ten points (twenty points for Division B schools), and the top team test score from each school will contribute to that school's overall point total. *Calculators are allowed on this event.*
SPRINT ROUND: In this individual test, students will have sixty minutes to complete 30 multiple choice questions. Four points will be awarded for each correct answer, with one point deducted for each incorrect answer; no penalty will be assessed for skipping. The top six scores from each school (or three for Division B schools) will be averaged to calculate that school's sprint round score. If fewer than six (or three) students take the test, zeros will be assessed for the leftover slots. This is to encourage schools to bring more students and not limit participation to only the one or two top math students in the school. *Calculators are not allowed on this event.*
TARGET ROUND: This is an individual event consisting of eight questions, each worth 10 points. Questions will be given out in pairs, and students will have ten minutes to complete each pair. Each question will be worth ten points. The top six scores from each school (or three for Division B schools) will be averaged to calculate that school's target round score. If fewer than six (or three) students take the test, zeros will be assessed for the leftover slots. This is to encourage schools to bring more students and not limit participation to only the one or two top math students in the school. *Calculators are allowed on this event.*
RELAYS: For this round, students will arrange themselves into teams of up to three. A relay consists of three questions, and each student will receive all three of these questions. The answer to question 1 is plugged in as TNYWR (The Number You Will Receive) to question 2 in order to solve question 2. This answer is then plugged in as TNYWR to question 3 in order to solve question 3. All three answers are integers in the range [0,99], and all three answers are written on the answer sheet, which is turned in at the end of three minutes. A correct answer for question 1 is worth 2 points; a correct answer for question 2 is worth 3 points; and a correct answer for question 3 is worth 5 points. Thus each team can accumulate up to 10 points on each relay. There will be five relays in the relay round, and the top two teams from a school count toward the school's total. *Calculators are NOT allowed on this event.*
POWER QUESTION: This is a multi-part, proof-oriented question that will test the students' higher-level mathematical reasoning skills. The power question is a team event in which groups of up to six (or three for Division B schools) work for one hour to produce a single multi-page answer. Scores for this event will be out of 100. Please note: due to time constraints the power question will not be offered as an event at local contests but will be administered at the state level. However, one power question will be made available to member schools as part of the mail-in qualifying round that coaches may administer in their own schools. *Calculators are allowed on this event.*
SWEEPSTAKES: A school's sweepstakes total is computed by adding its scores for each event. There are a maximum of 100 points available in the team test (the score of the highest scoring team from the school), 120 in the sprint round (average of the top six (or three) highest scores), 80 in the target round (average of the top six (or three) highest scores), and 100 in the relays (sum of the two (or one, doubled) highest scoring relay teams from the school), for a total of 400. At the state meet, the power question will be a competition event, so the maximum sweepstakes total will be 500.

More information about this year's contest sites and how you can participate can be found at https://moctm.org/mctm-math-contest/mctm-high-school-math-contest.html and http://mathleague.org/hs.php

Sample tests for these events can be found at http://mathleague.org/freetests.php .

Mathleague policies about homeschooled children, state qualification, and more can be found at http://mathleague.org/hsrules.php .

Solutions

Target (2002, 4th Qualifier, Round 4)

7. 1344
8. $\frac{1}{20}$

Relay (2001, 3rd Qualifier, Round 1)

(1-1) 81
(1-2) 5
(1-3) 80

Sprint (2001, 6th Qualifier)

1. B
2. A
3. A
4. C
5. E (215)
6. C
7. E (13)
8. B
9. D
10. D
11. C
12. D
13. C
14. D
15. D
16. C
17. B
18. C
19. B
20. B
21. C
22. A
23. D
24. A
25. E (1473)
26. E (10)
27. D
28. A
29. A
30. E ($149\pi/150$)

Team Round Solutions (L)

1. Work backwards. Before going into the last shop she had \$1024, before the lottery she had \$512, then \$1536, \$768, ... We can easily prove by induction that if she ran out of money after n shops, $0 \leq n \leq 10$, she must have started with $1024 - 2^{10-n}$ dollars.
Ans: **1023**

2. We know $b = 5 - a$, so $a(5-a) = 5$, or $a^2 - 5a + 5 = 0$. Now apply the quadratic formula and take the larger root.
Ans: $\mathbf{\frac{5+\sqrt{5}}{2}}$

3. The fraction of the original angle is $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - + \cdots$. This is just a geometric series with first term 1/2 and ratio -1/2, so the sum is 1/3.
Ans: **20°**

4. Solution 1: 123450789 leaves a remainder of 7 when divided by 91, and 1000 leaves a remainder of 90, or -1, so adding 7 multiples of 1000 will give us a multiple of 91.

Solution 2: For those who don't like long division, there is a quicker way. First notice that $91 = 7 \cdot 13$, and $7 \cdot 11 \cdot 13 = 1001$. Observe that $12345n789 = 123 \cdot 1001000 + 45n \cdot 1001 - 123 \cdot 1001 + 123 - 45n + 789$ It follows that 91 will divide $12345n789$ iff 91 divides $123 - 45n + 789 = 462 - n$. The number 462 is divisible by 7 and leaves a remainder of 7 when divided by 13.
Ans: **7**

5. By symmetry, it will suffice to consider one quarter of the dartboard, which is a square of side length 1. Therefore the probability is the area of the desired region in this square. The desired region is the part of the circle of radius 1 centered at a corner that is closer to the opposite corner. The points closer to the opposite corner are those that are on the other side of the diagonal through the other two corners, so the desired region is a quarter of a circle of radius 1 minus a right triangle with legs of length 1.
Ans: $\mathbf{\frac{\pi-2}{4}}$

6. Solution 1: Without loss of generality let $x > y$. We must have z^3 expressible as the sum of two squares, and this first happens when $z = 5$. Then x and y can be 10 and 5 or 11 and 2. If $z > 5$ then $z \geq 10$ for z^3 to be a sum of two distinct squares, so $x^2 > 500$, $x > 22$, so $x + y + z > 32$. Thus the smallest possible value of $x + y + z$ is $11 + 2 + 5 = 18$.

Solution 2: If $z > 5$, then $z \geq 6$, so $z^3 \geq 216$. Now $x^2 + y^2 \geq 216$, so $x \geq 11$ and $y \geq 1$, thus $x + y + z \geq 18$. Since $x = 11, y = 1, z = 6$ does not work, we must have $x + y + z > 18$, and the solution given is the best possible.
Ans: **18**

7. $\cos n\theta$ is the real part of $e^{in\theta}$, so the sum is the real part of $\sum_{n=0}^{\infty} \frac{e^{in\theta}}{2^n}$. This is a geometric series with initial term 1 and ratio $\frac{e^{i\theta}}{2}$, so its sum is $\frac{1}{1-e^{i\theta}/2}$. We are given $\cos\theta = \frac{1}{5}$, so $\sin\theta = \pm\frac{2\sqrt{6}}{5}$. Thus the sum is $\frac{10}{10-1\mp 2i\sqrt{6}} = \frac{90\pm 20i\sqrt{6}}{105}$, and we want the real part.
Ans: $\mathbf{\frac{6}{7}}$

8. There are 90 two-digit integers. So there are 8100 possible choices of two of them. We can get a multiple of 3 with remainders 0+0, 1+2, or 2+1, and there are 30 choices for each remainder.
Ans: $\mathbf{\frac{1}{3}}$

9. $\frac{25+16x^4}{x^2} = \frac{25}{x^2}+16x^2$. The the AM-GM inequality, this sum is at least as large as $2\sqrt{\frac{25}{x^2}\cdot 16x^2} = 40$, and the minimum is achieved when $\frac{25}{x^2} = 16x^2$.
Ans: $\mathbf{\frac{\sqrt{5}}{2}}$

10. We re-factor as $(x+1)(x-3)(x+2)(x-4)(x^2-2x-1)+24$, or $(x^2-2x-3)(x^2-2x-8)(x^2-2x-1)+24$, and this becomes $(y-4)(y-9)(y-2)+24$ where $y=(x-1)^2$. Now, $(y-4)(y-9)(y-2)+24 = (y-8)(y-6)(y-1)$, so y is 1, 6, or 8.
Ans: $\mathbf{0, 2, 1\pm\sqrt{6}, 1\pm 2\sqrt{2}}$.