1 Complex Numbers

For this worksheet, we denote the set of all real numbers as $\mathbb{R}$. For a quadratic polynomial $ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$, we can find the roots using the quadratic equation.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

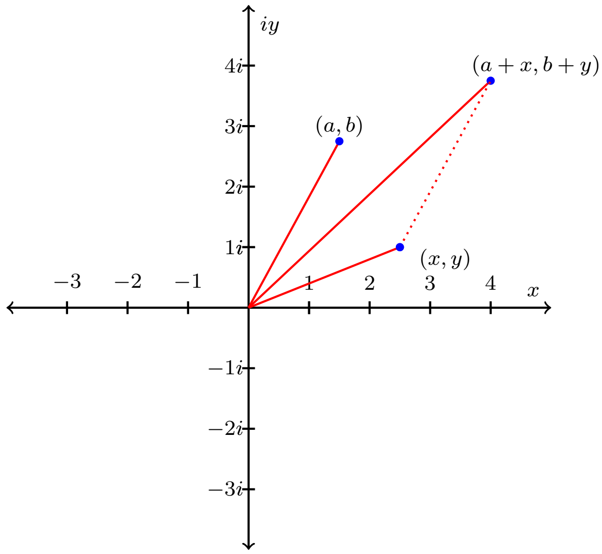
Classically an issue arose when $b^2 - 4ac < 0$ since the square root of a negative real number was undefined. Here the idea of a complex number was born, we define the imaginary number i,

$$i := \sqrt{-1}.$$

So now we can write numbers such as $\sqrt{-16} = 4i$ or $\sqrt{-48} = 4i\sqrt{3}$. These numbers and all others of the form bi where b is a real number (i.e. $b \in \mathbb{R}$) are called purely imaginary numbers. From this idea, we consider numbers of the form $z = x + iy$ where $x, y \in \mathbb{R}$. These numbers are known as complex numbers, and the space of all complex numbers will be denoted $\mathbb{C}$. Here are some operations and properties of complex numbers for $z = x + iy, w = a + bi \in \mathbb{C}$ and $c \in \mathbb{R}$

1. The conjugate of a complex number: $z = x + iy$, as $\overline{z} := x - iy$
2. The modulus of a complex number: $z = x + iy$, as $|z| := \sqrt{x^2 + y^2}$
3. Addition of Complex Numbers: $z + w = (x + iy) + (a + bi) := (x + a) + (b + y)i$
4. Scalar Multiplication: $cz = c(x + iy) = cx + cyi$
5. Multiplication of Complex Numbers: $zw = z \cdot w = (x + iy)(a + bi) := ax + bxi + ayi + byi^2 = (ax - by) + (bx + ay)i$. Notice we essentially "foiled" the two complex numbers and used $i^2 = -1$
6. $\overline{z + w} = \overline{z} + \overline{w}$
7. $\overline{zw} = \overline{z} \cdot \overline{w}$
8. $|zw| = |z| \cdot |w|$
9. NOTE: $|z + w| \leq |z| + |w|$ with equality when $z = \lambda w$ for some $\lambda \in \mathbb{R}$

If we consider a complex numbers $z = x+iy$ and $w = a+bi$ as and ordered pair (x, y) and (a, b) respectively then we can get a graphical representation of z, w, and $z + w$ in the $xy - plane$



If we use some trigonometry and polar coordinates,we can write $z = x + iy$ as $rcos(\theta) + irsin(\theta)$ where $r = |z|$ and θ is the angle z makes with the positive $x - axis$. If $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$, then $\theta = arctan(\frac{y}{x})$.
Euler's Formula: For a complex number $z = x + iy$ and r, θ defined above, $rcos(\theta) + irsin(\theta) = re^{i\theta}$ for k an integer.
Note In practice when using Euler's formula θ is taken in the range $0 \leq \theta < 2\pi$. But it is helpful to keep in mind that since $sin(\theta)$ and $cos(\theta)$ are 2π-periodic we can write $z = re^{i\theta} = re^{i\theta+2\pi ik}$ when k is an integer.
Examples

1. $z = 3i$ then $\theta = \frac{\pi}{2}$ and $r = 3$ so we can write z as $3cos(\frac{\pi}{2}) + 3isin(\frac{\pi}{2})$ and $3e^{\frac{i\pi}{2}} = 3e^{\frac{i5\pi}{2}}$
2. $z = -2\sqrt{3} - 2i$ then $\theta = \frac{7\pi}{6}$ and $r = 4$ so we can write z as $4cos(\frac{7\pi}{6}) + 4isin(\frac{7\pi}{6})$ and $4e^{\frac{7\pi i}{6}}$

2 Convolution

We now define the convolution operator on two finite sets of complex numbers $\mathbf{z} = \{z_1, z_2, ..., z_m\}$ and $\mathbf{w} = \{w_1, w_2, ..., w_n\}$ as,

$$\mathbf{z} * \mathbf{w} := \{(\mathbf{z} * \mathbf{w})_1, (\mathbf{z} * \mathbf{w})_2, ..., (\mathbf{z} * \mathbf{w})_{m+n-1}\}$$

where,

$$(\mathbf{z} * \mathbf{w})_i = \sum_{j=1}^{m} z_j w_{i-j}$$

for,

$$1 \leq i \leq m + n - 1$$

and, z_i and $w_j = 0$ if i, j are not in their respective ranges. This may seem like an unusual operation, but it arises naturally in the coefficients of the product of multiplied polynomials.
For polynomials

$$p(x) = a_m x^m + ... + a_1 x + a_0$$

and,

$$q(x) = b_n x^n + ... b_1 x + b_0$$

we can write,

$$p(x)q(x) = \sum_{i=0}^{m+n} \sum_{j=0}^{m} a_j b_{i-j} x^i$$

Example 1 Let's use convolution to verify that

$$(1 + 2x - x^3)(3 - 4x + x^2) = 3 + 2x - 7x^2 - x^3 + 4x^4 - x^5$$

We begin by calculating $\sum_{j=0}^{m} a_j b_{i-j}$ for $i \in \{1, 2, 3, 4, 5\}$

1. $(i = 0)$ $a_0 b_0 = (1)(3) = 3$

2. $(i = 1)$ $a_0 b_1 + a_1 b_0 = (1)(-4) + (2)(3) = 2$

3. $(i = 2)$ $a_0 b_2 + a_1 b_1 + a_2 b_0 = (1)(1) + (2)(-4) + (0)(3) = -7$

4. $(i = 3)$ $a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = (1)(0) + (2)(1) + (0)(-4) + (-1)(3) = -1$

5. $(i = 4)$ the only nonzero terms here are $a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b_0 = (1)(0) + (2)(0) + (0)(0) + (-1)(-4) + (0)(3) = 4$

6. $(i = 5)$ the only nonzero terms here are $a_0 b_5 + a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1 + a_5 b_0 = (1)(0) + (2)(0) + (0)(0) + (-1)(1) + (0)(-4) + (0)(3) = -1$

We see now that each value calculated matches the answer given above,so we are done.

3 The Discrete Fourier Transform

For a finite set, sequence, of complex numbers $\mathbf{z} = \{z_1, z_2, ..., z_m\}$ we define the Discrete Fourier Transform, DFT, as,

$$\mathcal{F}(\mathbf{z}) := \hat{\mathbf{z}} = \{\hat{z}_1, \hat{z}_2, ..., \hat{z}_m\}$$

with,

$$\hat{z}_k := \sum_{l=1}^{m} z_l e^{\frac{-2\pi i}{m}kl}$$

and its inverse as,

$$\mathcal{F}^{-1}(\hat{\mathbf{z}}) := \mathbf{z}$$

with,

$$z_n := \frac{1}{m}\sum_{k=1}^{m} \hat{z}_k e^{\frac{2\pi i}{m}kn}$$

Let $\mathbf{z} = \{z_1, z_2, ..., z_m\}$ and $\mathbf{w} = \{w_1, w_2, ..., w_m\}$and $c, \lambda \in \mathbb{C}$. Below are needed properties of the DFT:

1. $\mathcal{F}(c\mathbf{z} + \lambda\mathbf{w}) = c\mathcal{F}(\mathbf{z}) + \lambda\mathcal{F}(\mathbf{w})$ where $\mathbf{z} + \mathbf{w} = \{z_1 + w_1, z_2 + w_2, ..., z_m + w_m\}$
2. **Convolution Theorem** $\mathcal{F}(\mathbf{z} * \mathbf{w}) = \hat{\mathbf{z}} \cdot \hat{\mathbf{w}}$

Example Let's see why property 2 is true.

$$\begin{aligned}
\mathcal{F}((\mathbf{z} * \mathbf{w})_k) &= \sum_{l=1}^{m} (\mathbf{z} * \mathbf{w})_l e^{\frac{-2\pi i}{m}kl} \\
&= \sum_{l=1}^{m}\sum_{j=1}^{m} z_j w_{l-j} e^{\frac{-2\pi i}{m}kl} \\
&= \sum_{j=1}^{m} z_j \sum_{l=1}^{m} w_{l-j} e^{\frac{-2\pi i}{m}kl} \\
&= \sum_{j=1}^{m} z_j e^{\frac{-2\pi i}{m}kj} \hat{w}_k = \hat{z}_k \hat{w}_k
\end{aligned}$$

Since it holds true for the k^{th} component the equality holds for all entries.

4 Problems

Problem 1. (Euler's Identity): Write -1 in the form $re^{i\theta}$ to recover this famous identity.

Problem 2. (De Moivre's Theorem) Explain why $(rcos(\theta) + irsin(\theta))^n = r^n cos(n\theta) + ir^n sin(n\theta)$ and use this fact to prove the double angle trig identities for $sin(2\theta)$ and $cos(2\theta)$

Problem 3. Using polar form, $re^{i\theta}$, calculate the 3 values that satisfy $x^3 = 1$. Check by using the difference of cubes formulas.

Problem 4. Calculate i^{2019}, $\sqrt{i}$, and when $k = 0, i^i$

Problem 5. Using convolution calculate $(3 - 2x + x^2)(x - x^3)$

Problem 6. Stereographic Projection Imagine the sphere $x^2 + y^2 + z^2 = 1$. Starting at the North Pole (the point $(0, 0, 1)$), find a formula that maps every point on the sphere except $(0, 0, 1)$ uniquely to a point in the xy-plane.

Problem 7. Use the same idea as in problem 2 to find the formulas for $sin(3\theta)$ and $cos(3\theta)$. Now use Pascal's triangle to find a a formula for $sin(n\theta)$ and $cos(n\theta)$ when n is a natural numbers.

Problem 8. The quadratic equation is derived by setting $ax^2 + bx + c = 0$ dividing by a then completing square. An explicit formula for the roots of a cubic polynomial exists. For simplicity, use $x^3 + ax^2 + bx + c$ to find an explicit formula for the roots of this cubic polynomial. (Hint: Take the substitution $x = y - \frac{a}{3}$)

Problem 9. Prove the Triangle Inequality. (Start by considering what $z\bar{z} =?$ and using this for $|z + w|$)

Problem 10. Use $\sum_{n=0}^{N} z^n = \frac{z^{N+1}-1}{z-1}$ to calculate $\sum_{n=0}^{N} cos(n\theta)$

Problem 11. Use the DFT to calculate $\mathcal{F}(\{1, 0, 0, 0\}), \mathcal{F}(\{0, 1, 0, 0\}), \mathcal{F}(\{0, 0, 1, 0\})$, and $\mathcal{F}(\{0, 0, 0, 1\})$.

Problem 12. Use the Convolution Theorem and Inverse DFT to calculate $(3 - 2x + x^2)(x - x^3)$.

5 Further Reading

1. The Fundamental Theorem of Algebra
2. The Quartic Equation for the roots of a polynomial of the form $ax^4 + bx^3 + cx^2 + dx + e$
3. Why no formulas exist for polynomials of degree 5 or higher
4. The Convolution of Functions
5. The Fast Fourier Transform