

Fun with Paradoxes

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Introduction

What does a paradox mean to you? A simple definition of a paradox is a statement that seems to contradict itself. This contradiction can be very confusing! Paradoxes can come in many flavors. We shall see that paradoxes can arise in all areas of mathematics. Some paradoxes come about because of a lack of understanding or intuition. The conclusion may seem contradictory at first, but actually all the reasoning is logically sound. Other paradoxes either indicate poor mathematical reasoning or some deeper problem with underlying logical or foundational issues.

Question 1. Give an example of a paradox (it doesn't have to be mathematical).

1 A lighthearted example

The following example of a paradox maybe doesn't represent a serious logical problem like some of the others we will study, but it's fun!

Example 2. Consider the set of all positive integers. Are all these numbers interesting? Well, turns out they have to be! Suppose to the contrary that not all positive integers are interesting. Then there exists a least number, call it m , that is not interesting. What is contradictory about this?

Perhaps this seems "paradoxical" because the claim that all numbers are interesting on its face appears to be unreasonable. The proof also seems like a sleight of hand- it doesn't

really tend to convince most people! But notice that everything in this example hinges on the word “interesting.” And that word can mean different things to different people!

2 Logical Paradoxes

Some paradoxes arise seemingly because of the “structure” of logic itself. For example, it is impossible to assign a truth value to certain sentences. Most sentences, particularly in the context of mathematics can be assigned a definite truth value. For instance, the sentence “there exists a number whose square is four” is true, while the sentence “there exists precisely one number whose square is four is false” because there are precisely two numbers which square to four. However, the following example illustrates there are sentences for which this fails.

Example 3. Consider the sentence “this sentence is false.” Explain why this sentence cannot be assigned a truth value without leading to a logical contradiction.

There are even more complicated examples, such as the following statement which is known as Curry’s paradox.

Example 4 (Curry’s Paradox). Consider the sentence: If this sentence is true, then Germany borders China.

Question 5. Note that logical implications are written in the form “if A , then B .” What are A and B in this case?

Question 6. Assume A . Why does this automatically imply that B is true?

Question 7. The previous question implies that the sentence (as a whole) is true. Why does this lead to a paradox?

In some instances, paradoxes can arise because of inherent problems with definitions or *axioms*. *Axioms* are mathematical statements which we assume to be true without proof. You can think of axioms as a foundation for a house or building. Oftentimes, we say that axioms are "obvious" or "self evident," but sometimes figuring out which axioms are necessary to have a consistent theory is more complicated than it would seem. For example, mathematicians for a while believed that defining the word *set* as a collection of objects was okay. This seems like a perfectly reasonable definition! In fact, clever mathematicians and logicians realized that there were problems with this definition as the following example illustrates:

Example 8 (Russell's paradox). Let S be the set of all sets that do not contain themselves as an element. Does S contain itself? Why does this lead to a contradiction either way?

If this is a bit too abstract, consider the following example, which is really just a rewording of Russell's paradox:

Example 9. Suppose there is a barber in a town who shaves all the men who do not shave themselves. Who shaves the barber? Explain the paradox.

3 Problems with Proofs

In some cases, paradoxes or absurd statements can come about because we have actually made a mistake in our reasoning. Thus, the paradox can be resolved by identifying the mistake in our reasoning. The following example is a classic. Before we can proceed though, we need to review the proof technique of induction.

Induction is usually used to prove that a statement holds for all positive integers. The steps are as follows:

1. Prove that the statement holds for a *base case* (usually this corresponds to $n = 0$ or $n = 1$, but not always).
2. Assume the claim holds for some positive integer n . (this is called the induction hypothesis).
3. Show this implies that the claim holds for the integer $n + 1$ (this is called the induction step).

We illustrate this procedure by proving that, for all integers $n \geq 1$:

$$\sum_{j=1}^n j = \frac{n(n+1)}{2}. \quad (*)$$

Proof.

To see the base case is true, simply plug $n = 1$ into the formula. We get

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1 = \sum_{j=1}^1 j,$$

so the base case holds.

Next, suppose that for some integer n , we have

$$\sum_{j=1}^n j = \frac{n(n+1)}{2}.$$

Assuming the above equation is true, we compute:

$$\sum_{j=1}^{n+1} j = \sum_{j=1}^n j + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2n + 2}{2} = \frac{(n+1)(n+2)}{2}.$$

This string of equalities shows that the statement (*) holds for the integer $n + 1$ (just substitute $n + 1$ in for n into (*)). So, by induction, the formula (*) is true for all positive integers n . □

Question 10. Identify the exact step above where we used the induction hypothesis.

We now proceed to “prove” by induction that all horses are the same color. Precisely, we prove that given any group of n horses, all the horses are the same color. Obviously, the conclusion is absurd, so that tells you there must be a problem with the proof! See if you can identify exactly where the error is.

Example 11 (All horses are the same color). *Proof.* The base case $n = 1$ is easy, since obviously a single horse is a single color. Assume that given a group of n horses, all of them have the same color. Now suppose we have a group of $n + 1$ horses. Excluding one horse, we get a group of n horses. By our induction hypothesis, all these horses have the same color, which we might as well assume is brown (it could be black or white or whatever, but it doesn’t matter). Now exclude a different horse from the group. This creates a different group of n horses, and by the induction hypothesis they must all have the same color as well. Since the horses that are in both groups only have one color, this shows that the horses in the second group are colored brown. Thus, all of the horses are brown. \square

Question 12. Identify exactly where the error is in the preceding proof.

4 A Paradox of Probability

The following scenario leads to a conclusion that may seem paradoxical, but actually the mathematical reasoning is sound and the paradox is not telling us there is something wrong with our mathematics. The following is a version of what is famously called the Monty Hall Problem. The answer is very surprising!

Example 13 (Monty Hall Problem). Suppose a contestant is on a game show. The contestant has to choose one of three doors. Behind one door is a car, and behind the other two doors are goats. If the contestant picks the door with the car, he or she wins the car. Otherwise, he or she goes home empty handed. The contestant makes his or her choice of

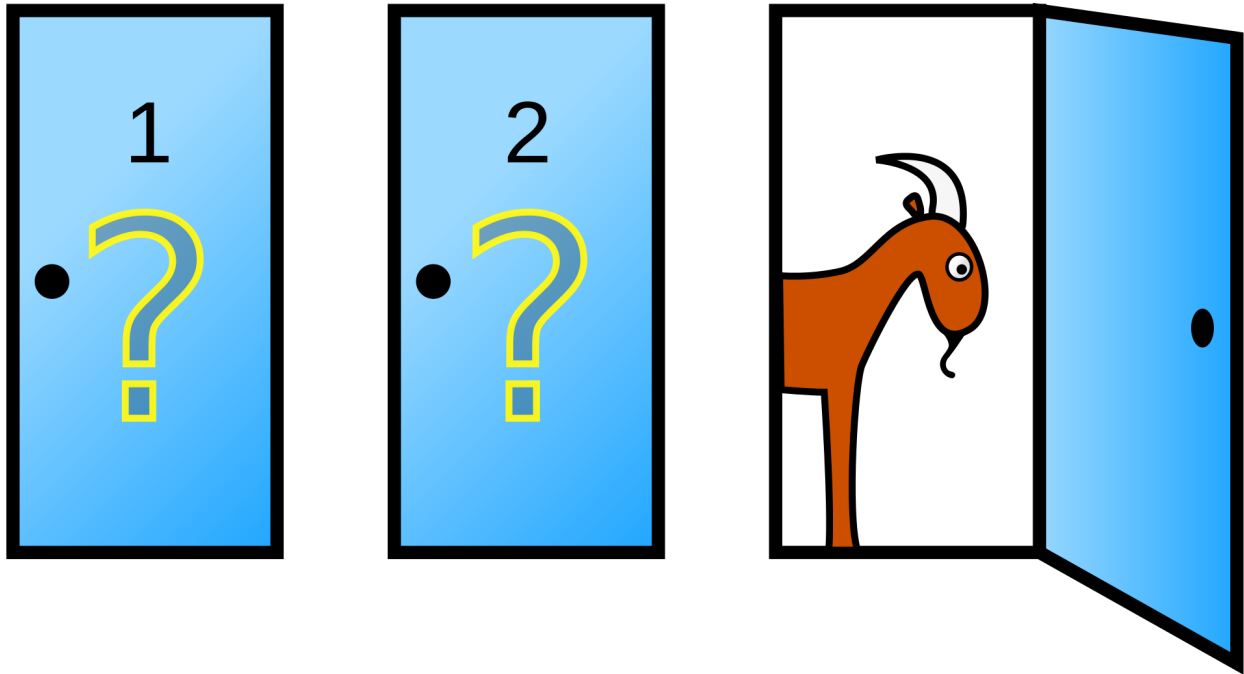


Figure 1: The Monty Hall Problem

door, and then the host opens one of the two unchosen doors that contains a goat. Assume that if both of the unchosen doors contain a goat, the host is equally likely to open either of them. Is it beneficial for the player to switch doors? Explain your answer and calculate the probabilities.

5 Paradoxes with Summation

We know how to add up finitely many numbers, but things become much more complicated when we turn to adding infinitely many numbers. Loosely speaking, a series is an infinite sum, and one of the most famous series is the following:

$$\sum_{n=1}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

So we begin by taking 1, then subtracting 1, then adding 1, then subtracting 1, and so on indefinitely. The problem is to determine what we obtain when we do this procedure.

Question 14. Show that by grouping terms appropriately, you can argue the infinite sum is 1.

Question 15. Show that by grouping terms appropriately, you can argue the infinite sum is 0.

Since we got two different answers for the sum depending on the grouping of terms, something must be wrong. In fact, mathematicians realized that they had to develop a more rigorous definition of an infinite sum than simply “adding up infinitely many numbers.” More precisely, they developed the notion of *convergence* or *divergence* of a sum. Loosely speaking, a sum converges if when we keep adding up terms sequentially, the numbers we get at each step approach a single number. This number is then the sum of the series. In the previous example, the sum does not converge, it instead diverges. This is because the *partial sums* switch back and forth between 1 and 0 when we add up the numbers sequentially.

There is a specific class of series that we know converge. They are called geometric series. First, consider a finite geometric sum with common ratio r :

$$\sum_{n=1}^N r^n = 1 + r + r^2 + \cdots + r^N.$$

Question 16. Compute $(1 - r) \sum_{n=1}^N r^n$ and notice it simplifies very nicely. Then divide through by $(1 - r)$ to obtain a formula for a finite geometric sum.

If $|r| < 1$, it can be shown by taking a limit of these partial sums that

$$\sum_{n=1}^{\infty} r^n = \frac{1}{1 - r}.$$

Thus, in this case the infinite series converges and we actually have a formula for the value of the infinite sum.

Question 17 (For those with exposure to limits). Show that if $|r| < 1$, then

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N r^n = \frac{1}{1-r}.$$

This proves the formula for infinite geometric series.

The next exercise proves that $0.\bar{9} = 1$, a result that is surprising to most students and tends to be counter intuitive at first. Note when we write $0.\bar{9}$, we mean the repeating decimal $0.999999\dots$

Question 18. Write the repeating decimal $0.\bar{9}$ as a geometric series and use the formula to show that the sum of this series is 1.

6 Challenge Problems

Work on these problems only if you have finished all the others.

This first problem requires integral calculus, in particular volumes of solids of revolution and surface area, as well as improper integration. Ask for help if you are unfamiliar with these concepts.

This paradox involves Gabriel's horn, a mathematical object with finite volume and infinite surface area.

Question 19. Gabriel's horn is formed by rotating the (unbounded) region in the xy -plane between the x -axis and the curve $y = \frac{1}{x}$ for $x \geq 1$ about the x axis. Show using calculus that Gabriel's horn has finite volume but infinite surface area.

The next “paradox,” the existence of a non-measurable set, requires some background in a field of mathematics called *measure theory*. To keep things simple, we’ll only consider measures defined on \mathbb{R} . A *measure* μ is a function $\mathcal{A} \rightarrow [0, \infty]$, where \mathcal{A} is some collection of subsets of \mathbb{R} satisfying the following two properties:

1. $\mu(\emptyset) = 0$, where \emptyset denotes the empty set, or set with no elements.
2. If sets $A_1, A_2, \dots, A_j, \dots$ are pairwise disjoint (this means that $A_i \cap A_j = \emptyset$ when $i \neq j$), then we have

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$$

This second property is referred to as *countable additivity*. It basically says that when we decompose a set into pieces and add up the measure of the pieces, we get the measure of the set.

You should think of a measure as a function that somehow quantifies the “size” of a set. It turns out that the measure most commonly used on the real line, Lebesgue measure, corresponds on simple sets to what we would think of the “length” of the set. However, Lebesgue measure allows us to quantify the size of sets much more diverse than just intervals or finite unions and intersections of intervals (where we could easily just add up the lengths). There is a serious drawback in measure theory though- it is impossible to assign a meaningful size or measure to all sets! We will see this with the next example.

Example 20. Suppose μ is a measure that is well-defined on ALL subsets of \mathbb{R} so that μ coincides with our ordinary notion of length on intervals; that is, $\mu([a, b]) = b - a$. We will show that such a measure does not exist through a proof by contradiction. In particular, we construct a set F and show that the properties of μ lead to contradictory information about F . This basically shows that it is impossible to assign a meaningful size or length to all subsets of \mathbb{R} . The set F is an example of what we call a *non-measurable set*, which may seem strange or paradoxical.

We construct the set F as follows. First, define an equivalence relation \sim on $[0, 1]$ as follows: $x \sim y$ if $x - y$ is rational.

Question 21. Show that \sim is an equivalence relation (ask for help if you don't know what an equivalence relation is).

Now define the set F as follows: choose exactly one element of each equivalence class defined above. The resulting collection of real numbers is the set F . Pretty strange way to define a set, huh?

Now, for a real number q , define $F + q = \{f + q : f \in F\}$. Thus, $F + q$ represents a “translate” of the set F by q units. It can be shown (we omit the details) that $\mu(F) = \mu(F + q)$ for any rational q .

Question 22. Explain why the sets $F + q$ are disjoint for distinct rational q .

Question 23. Show that $[0, 1] \subset \bigcup_{q \in [-1, 1] \cap \mathbb{Q}} F + q$.

Question 24. Explain why this implies $\mu(F) > 0$.

Question 25. Show that $\bigcup_{q \in [-1, 1] \cap \mathbb{Q}} F + q \subset [-1, 2]$.

Question 26. Show that this implies $\mu(F) = 0$ and leads to a contradiction.