Math Circle (Oct. 27th, 2019). Presenter: Haohua Deng
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Solving Combinatorial Problems by telling Stories

We are going to deal with some elementary combinatorial problems. By "telling Stories", I mean constructing a model which is easier to visualize than mathematical formulas, or describing the problem by some down to earth languages. Sometimes such ideas could be very useful and efficient.

Note: We denote $\binom{n}{k}$ as the # of ways for picking k out of n distinct objects, and we have $\binom{n}{k} = \binom{n}{n-k}$. Mathematically $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, but you don't need to use this formula in most parts of this worksheet.

Let us begin with some examples:

(1). Suppose we have a set of n elements. What is the total number of subsets of it? ($\emptyset$ and the set itself should be included.)

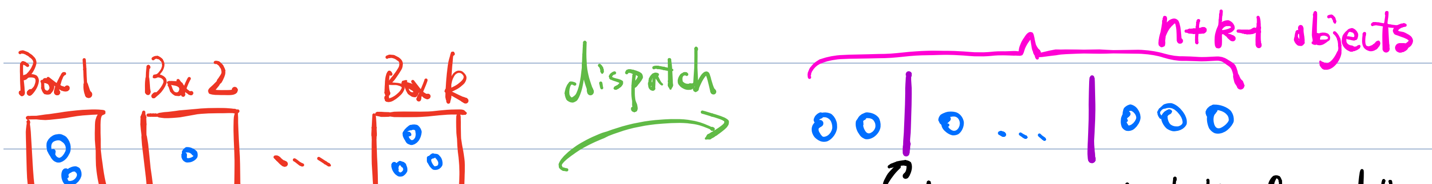
Bob is going to hold a party tonight, and he sends out invitations to n of his friends. For each one in the invitation list, he or she has exactly two options: Accept or Decline. Therefore, the subset of the set of invited person has exactly 2^n possibilities, which is exactly the # of subsets of a set with n objects.

Moreover, we get a formula: $2^n = \sum_{i=0}^{n} \binom{n}{i} = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$. ∎

Let us consider a more challenging one:

(2) Find the # of solutions for the equation: $x_1 + \dots + x_k = n$, where k, n are positive integers, and each x_i is a non-negative integer.

Now, consider $x_1, \dots, x_k$ as k boxes. You have n balls, and you need to put them in these boxes. How many ways for you to do that?



Add a bar to separate balls from different boxes.

We can see, each way to assign these balls is <u>uniquely</u> determined by a sequence of $n+k-1$ objects (which contains n balls & $k-1$ bars). Therefore, # of different such sequences = # of different ways to assign these balls = # of solutions of the above equation. For the # of different such sequences, we can see that it is just equal to # of ways to take $k-1$ slots out from the sequence for placing bars, thus the answer is $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$. ∎

(3) Prove the formula: (1) $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

(2) For $m < n$, $\sum_{k=0}^{n-m} \binom{n-k}{m} = \binom{n+1}{m+1}$ (Slightly harder).

How many ways to choose n out of $2n$ cookies? What if these $2n$ cookies are made up by n sweet cookies & n salty cookies? If you want to choose $m+1$ out of $n+1$ <u>labelled</u> cookies, among each set of $m+1$ cookies you choose, what about the one with smallest index?

It is time to take some exercises!

1. Prove the formulas: (1). $\binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell}\binom{n-\ell}{k-\ell}$ $(\ell<k<n)$

(2) $\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}$ (*)

2. How many ways to choose 2 subsets A, B from $\{1,2,\dots,n\}$ s.t. $A \subset B$? What if we want $A \subsetneqq B$? What if we want $A \neq \phi$? (*)

3. Suppose $\mathcal{A}$ is a set of subsets of $\{1,2,\dots,n\}$, and satisfies:

(1). Every member of $\mathcal{A}$ has exactly k elements.

(2) For $\forall\ i \in \{1,2,\dots,n\}$, i is contained in exactly r members of $\mathcal{A}$

How many elements (i.e. subsets of $\{1,2,\dots,n\}$) does $\mathcal{A}$ contain? (**)

4. How many monotone-increasing functions from $\{1,2,\dots,n\}$ to itself? (Monotone-increasing means $f(a) \leq f(b)$ whenever $a \leq b$). (**)

5. Prove that for any non-negative integers m, n, $\frac{(2m)!\,(2n)!}{(m+n)!\,m!\,n!}$ is always an integer. (***)

Hints / Solutions

1. (1). Among n students, we choose k of them to participate in sport games. Among these k selected students, ℓ of them want to play football and the remaining want to play basketball. How many possible situations?

2. A = Students with grade $\geq$A ; B = Students with grade $\geq$B. What is your grade? $\geq$A, $\geq$B but <A or <B ?

3. Suppose A has x elements. Image you are teaching n kids mathematics. You give them x problems, and you promise them that each one of them will receive a candy for each correct answer he/she provides. If each of them provides exactly r correct answers, and for each problem you receive exactly k correct answers. how many candies you should prepare for them?

4. There are n children labelled as $1, \dots, n$, and each of them has no more than n dollars. Moreover, the $(i+1)^{th}$ children has no less money than the i^{th} does. What if for each i, the i^{th} children receives $i-1$ dollars in addition?

5. I don't have such a story. I wish you have one (and I will be appreciated if you can tell me!)
If you know some tricks in number theory, then try to prove it by looking at the power of P in the dividend & divisor, where P is any prime number.