Math Circle - November 17, 2019

What's a derivative? What's differentiation?

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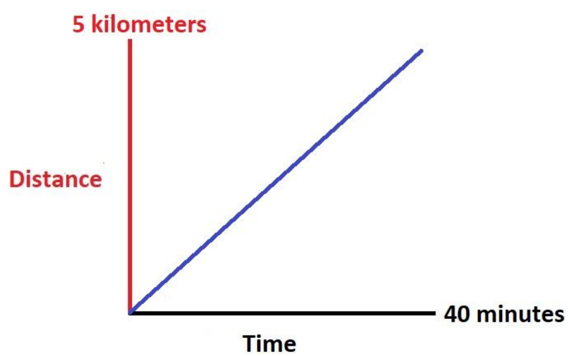
Today's goals are the following:

- Have a brief introduction to the concept of derivative.
- Learn how to differentiate a polynomial together with examples that, in the words of Dr. Thornton, make "physical sense".

Functions

We care about a *derivative* because it gives us information on "how fast something is changing" **(Lighthouse, 01/23/16)**. Now, the concept of derivative is strictly related with the concept of *function*. Let's start with an example.

Example: Suppose that you want to walk from your house to the house of your best friend, which is 5 kilometers from you, and it takes you 40 minutes. The question is: can you use a function to describe how much distance you traveled at each moment of that 40 minutes? The answer is a big... YES! The entry data is the time and the output data is the distance traveled. If you walk at a *constant* speed, then the graph of the function will look like...



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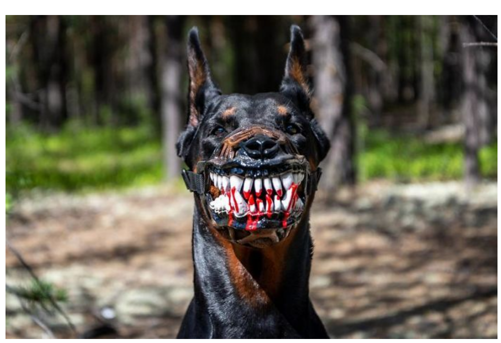
If you see a scary dog when walking there...



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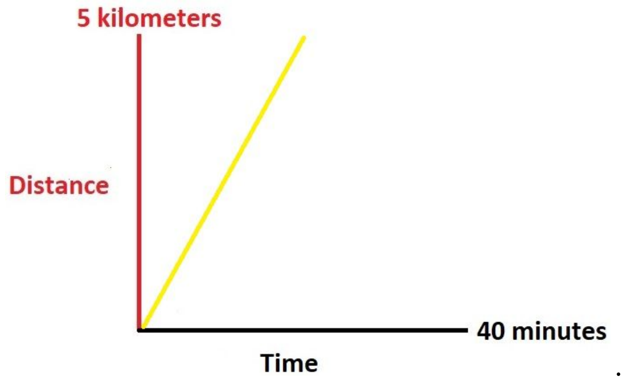
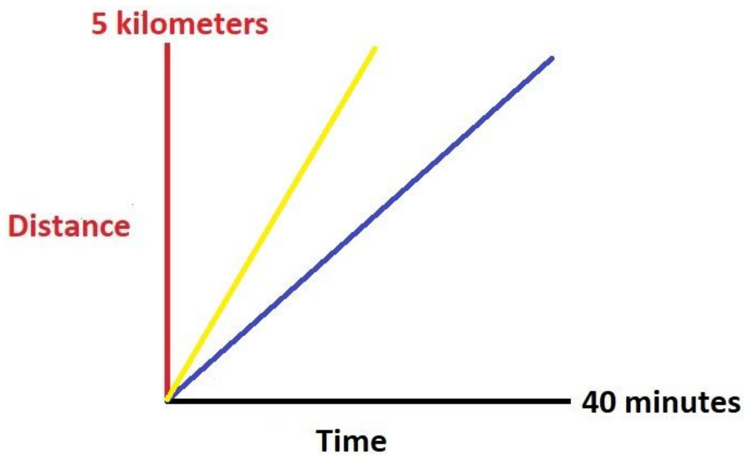
and start to walk faster, or run, but with *constant* speed, then the graph can, instead, look like...



That is, you traveled the *same* distance, but in *less* time. This is being shown by the *higher* slope. In other words, with respect to the graph (i.e., the line), *higher speed* means *more inclination* and *less speed* means *less inclination*.



On the other hand, if you have being *changing* speed, how will the graph looks like?

Suppose that there were slopes…

and sometimes you get distracted by playing nintendo switch...

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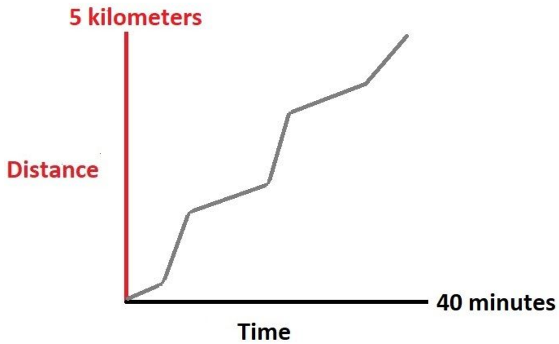
among others factors that were *changing* your speed. Then, the graph can look like...



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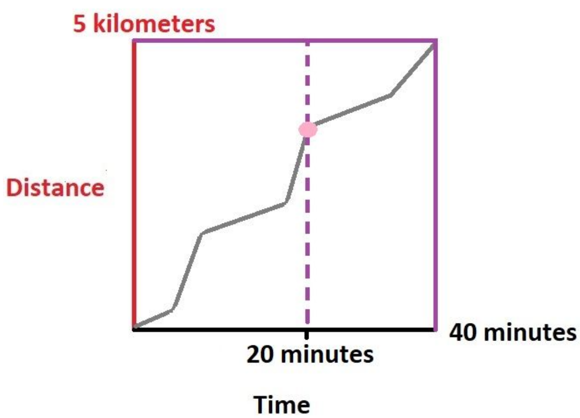
Thus, we end up with a graph containing *variations of inclination*. Here is when you start to care about the *derivative*, since it will helps you measure, at each point, how fast you were going.

What's a derivative?

Imagine you want to know what speed you had at 20 minutes. You know the distance traveled between minute 0 and minute 40, which is 5 kilometers, so that you can compute the average speed: 5/40=0.125. But, that's *not* going to be a *good* approximation.



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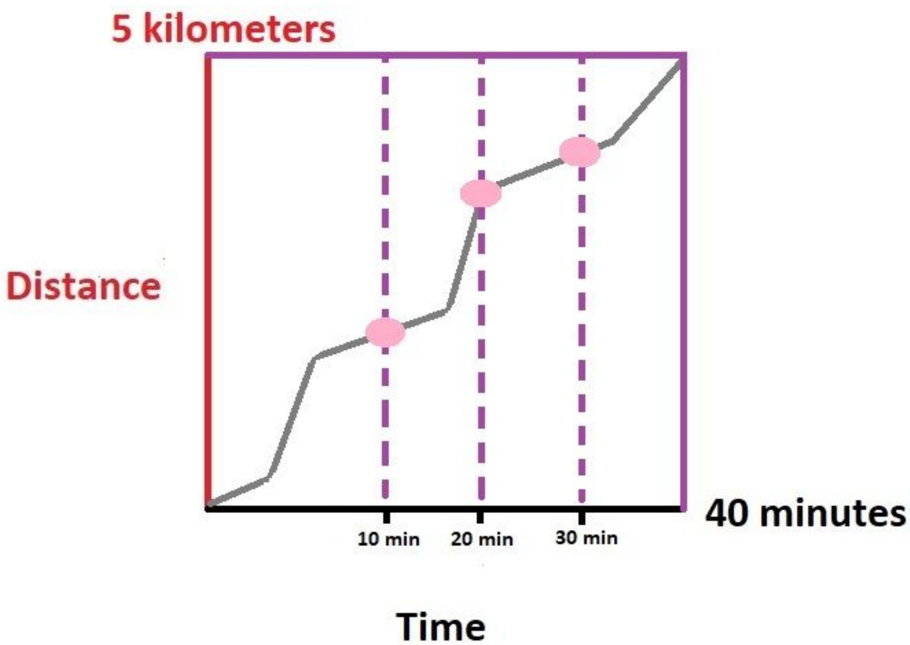
However, instead of taking 40 minutes, you could instead take 8 minutes, 1 minute, 10 seconds, 0.5 second, 1 tenth, 1 thousand, and so on.

The derivative is a tool that you use to “measure the limit of the change in position when the time interval tends to zero” **(Derivando, 4/17/16)**.

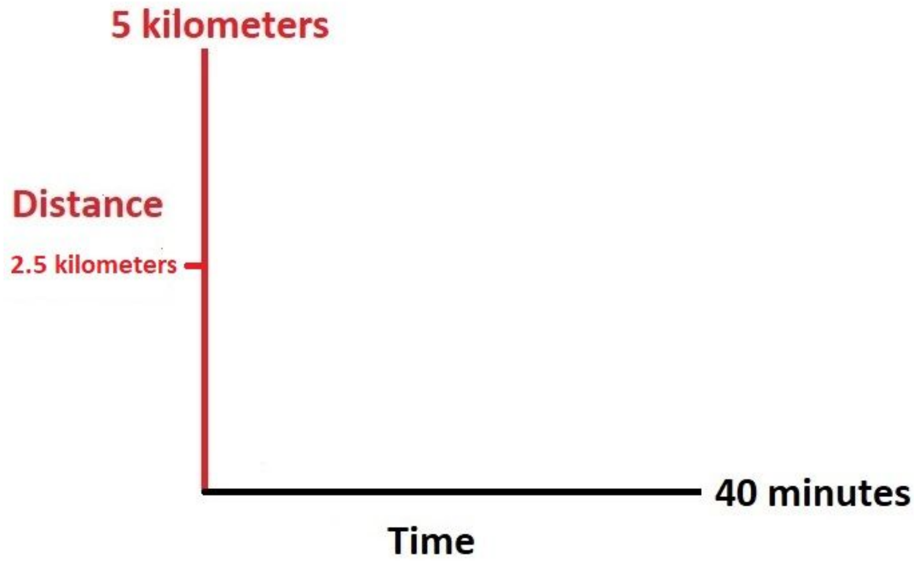


In other words, the function measures your traveled distance in terms of the time that has passed, and the derivative measures the speed at each point (that is, at each instant) of your travel.

First problem set

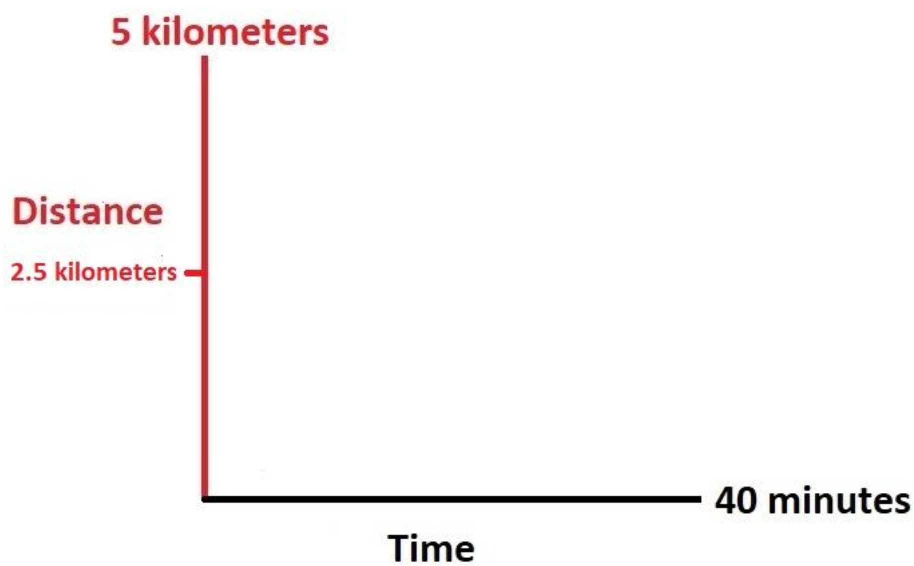
1. With respect to the previous example, how the graph will look like if you run at a constant speed since the beginning of your journey until 2.5 kilometers (escaping from the scary dog). Then, you walk at a constant speed until the house of your best friend (since you’re tired, and already escaped from the scary dog). Suppose, again, that the journey takes you 40 minutes.

   Answer (see next page):

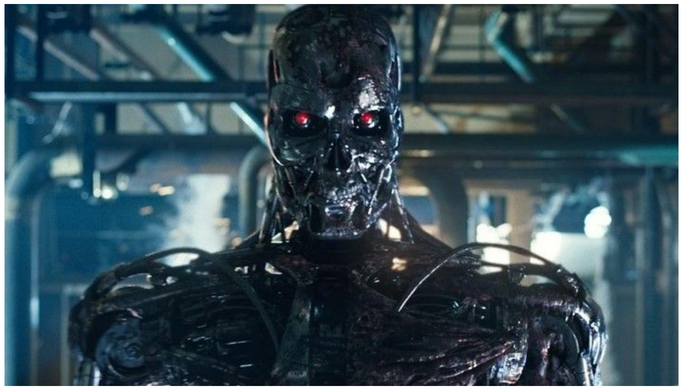


2. On the other hand, suppose you walk at a constant speed for the first 2.5 kilometers, since you are playing nintendo switch. Then, the scary dog suddenly appears, so that you run, but at a constant speed, during the next 2.5 kilometers (escaping from the scary dog) until you get to the house of your best friend. Suppose that, again, the journey takes you 40 minutes. How the graph will look like?

   Answer:

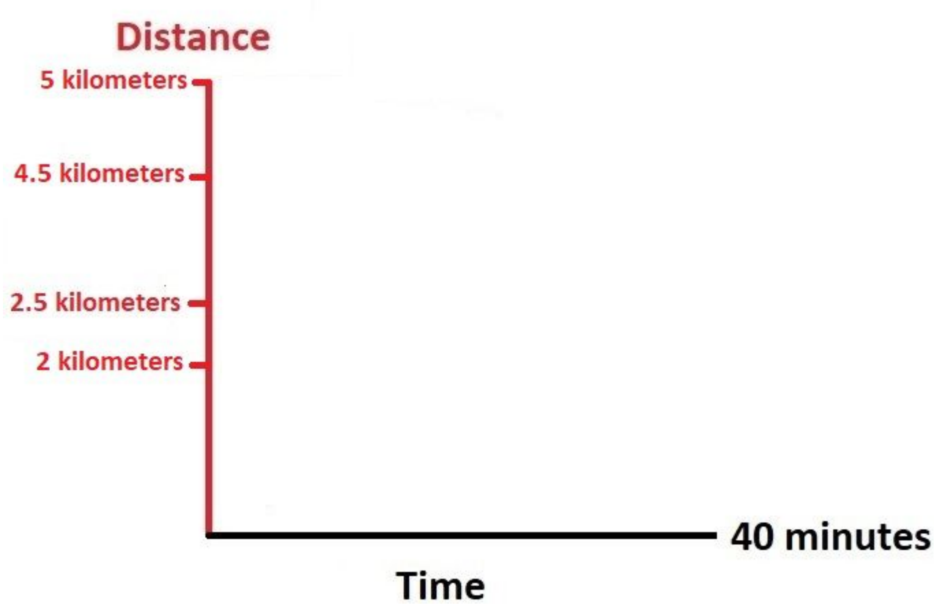


3. Again, suppose you see the scary dog at the beginning of your journey, so that you start to run at a constant speed from your house until 2 kilometers, where you escaped from the scary dog. Then, you walk, or run slower, at a constant speed until your traveled distance is 2.5 kilometers. Then, you start to run again at a constant speed, but the *same* speed as when escaping from the scary dog, since you see the following terminator…

.

But, when you get up to 4.5 kilometers, then you start to walk, or run slower, again at a constant speed (since you noticed that the terminator doesn't care about you because its objective is to find John Connor and Sarah Connor) until you arrive to the house of your best friend. Suppose, again, that the journey takes you 40 minutes. How the graph will look like?

Answer:



4. Let's change the situation for problem 3. and make it more interesting. Imagine that you see the scary dog since the beginning of your journey, so that you start to run at a constant speed from your house until 2 kilometers, where you escaped from the scary dog. Then, you walk or run slower at a constant speed until you achieve 3 kilometers, where you start to run at a constant speed, but *faster* than when escaping from the scary dog (since you see that the terminator in problem 3. starts to walk towards you). But, when you achieve 4 kilometers, you start to walk or run slower again (at a constant speed) until the end of your journey. (This because the terminator noticed that you are neither John Connor nor Sarah Connor, so it stopped chasing you.) Suppose now that the journey takes you *30* minutes. How the graph will look like?

   Answer:

5. Suppose that you decide to visit your best friend, but after being already 2 kilometers away from your house. Consider the following data to answer the following questions.

Time (in minutes)	Distance (in kilometers)
0	2
10	3
19.5	3.95
20.5	4.05
25	4.5
30	5

Find the following:

a. What's your average speed?

   Answer:

b. Now, compute the following average speeds:

   - average speed when considering the period of time from 10 minutes to 25 minutes.

     Answer:

   - average speed when considering the period of time from 19.5 minutes to 20.5 minutes.

     Answer:

c. Now, assume that the distance, denote it by $f(x)$, traveled at a time x is given by the equation: $f(x) = \frac{x}{10} + 2$. For example, note that $f(10) = 3$, which is equivalent to say that at 10 minutes, your traveled distance is 3 kilometers. (Note that this match with what we have in the previous table.)

What can you conclude about your speed during the whole journey?

Answer:

6. Finally, suppose that you decide to visit your best friend, but after being already *half a kilometer* away from your house. Now, you'll assume that the distance from your house to the house of your best friend is *5.5 kilometers*. Consider the following data to answer the following questions.

Time (in minutes)	Distance (in kilometers)
0	0.5
10	4/3
19.5	2.125
30.5	3.0416666...
45	4.25
60	5.5

Find the following:

a. What's your average speed?

Answer:

b. Now, compute the following average speeds:

- average speed when considering the period of time from 19.5 minutes to 30.5 minutes.

  Answer:

- average speed when considering the period of time from 30.5 minutes to 45 minutes.

  Answer:

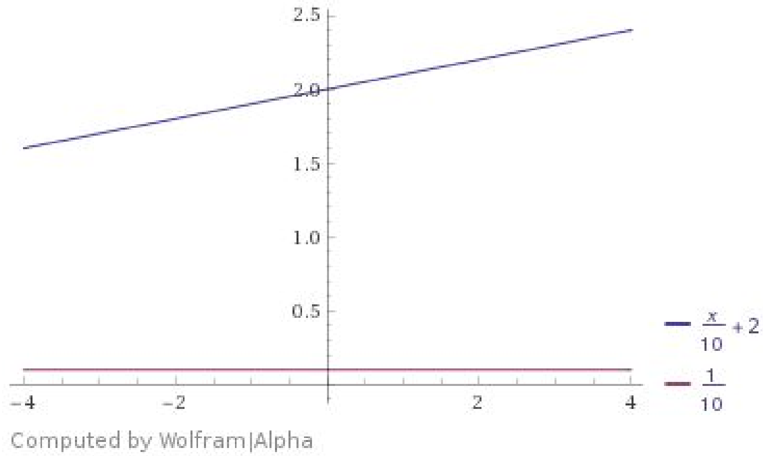
c. Now, assume that the distance, denote it by $f(x)$, traveled at a time x is given by the equation: $f(x) = \frac{x}{12} + \frac{1}{2}$. For example, note that $f(45) = \frac{45}{12} + \frac{1}{2} = 4.25$, which is equivalent to say that at 45 minutes, your traveled distance is 4.25 kilometers. (Note that this match with what we have in the previous table.)

   What can you conclude about your speed during the whole journey?

   Answer:

What's differentiation?

You usually express a function by way of a formula. For example, in the fifth problem you considered $f(x) = \frac{x}{10} + 2$. Then, you can obtain the derivative of f, $f'(x)$, by a mathematical operation called *differentiation*. In this case, you basically concluded already that $f'(x) = \frac{1}{10}$. There are basic rules like the ones you are going to learn today when computing the derivative of a polynomial function (like the previous one).



How to differentiate a polynomial?

Let's develop the idea of how to differentiate a polynomial by working with some problems.

Second problem set

7. Look at the following table, which contains polynomial functions with only one term, and its derivatives.

f(x)	*f'(x)*
…	…
x^5	$5x^4$
x^4	$4x^3$
x^3	$3x^2$
x^2	$2x^1 = 2x$
$x^1 = x$	$1x^0 = 1$

Identify pattern(s) in the previous table that helps you to conclude how you should proceed to differentiate $f(x) = x^n$. Then, write a formula for $f'(x)$.

Answer:

8. Look at the following table, which contains polynomial functions with only one term, and its derivatives.

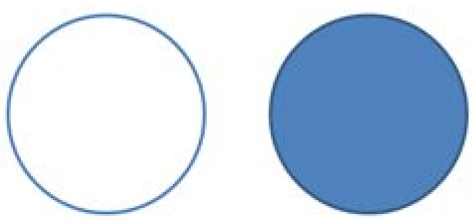
f(x)	*f'(x)*
...	...
$2x^5$	$2 \times 5x^4 = 10x^4$
$6x^4$	$6 \times 4x^3 = 24x^3$
$-5x^3$	$-5 \times 3x^2 = -15x^2$
$-\pi \times x^2$	$-\pi \times 2x^1 = -2\pi x$
$0.1 \times x^1 = 0.1x = \frac{x}{10}$	$0.1 \times 1x^0 = 0.1 = \frac{1}{10}$

Identify pattern(s) in the previous table that helps you to conclude how you should proceed to differentiate $f(x) = ax^n$, where a is any non-zero constant (number). Then, write a formula for $f'(x)$.

Answer:

Now, let's look at some examples in which the previous rule makes "physical sense".

9. Consider the following picture.



$2\pi r$ πr^2

Note that in the picture is given how you can obtain the circumference of the circle and how you can obtain the area of the disk.

a. If a disk has area of 4π. What's the circumference of the circle that surrounds it?

Answer:

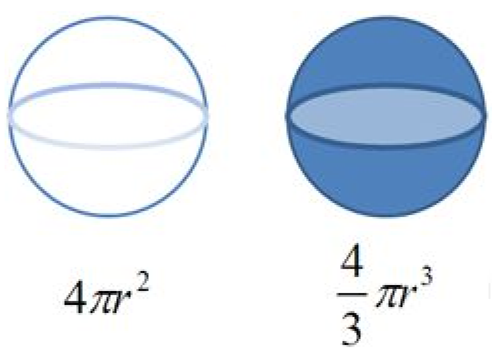
b. If a circle has a circumference of $\frac{2}{3}\pi$. What's the area of the disc it surrounds?

Answer:

c. Use your conclusion of problem 7. to explain how you can obtain the formula for the circumference of the circle from the formula of the area of the disk? *Hint*: Note that here *r* is used to denote the radius, but you can, instead, use *x* to denote the radius (or any other letter).

Answer:

10. Consider the following picture.



a. Note that in the picture is given how you can obtain the surface area of the sphere and how you can obtain the volume of the sphere. If a sphere has volume of 12π. What's its surface area?

Answer:

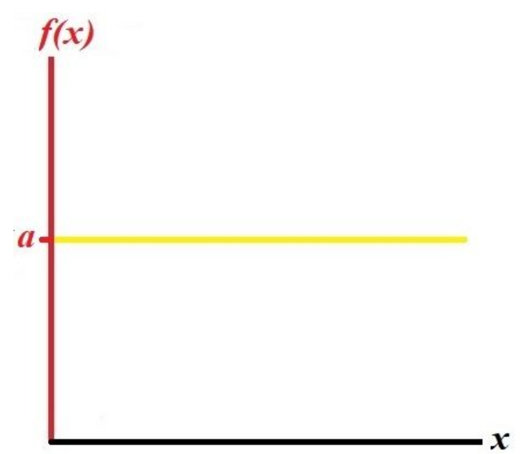
b. If a sphere has a surface area of 36π. What's its volume?

Answer:

c. Use your conclusion of problem 7. to explain how you can obtain the formula for the surface area of the sphere from the formula of the volume of the sphere?

Answer:

11. Suppose that $f(x) = a$ is a constant function. (That is, a is any constant (number).) Hence, we know that the graph of f is going to look like…



(Of course, a can be either zero or nonzero.)

What's $f'(x)$? Equivalently, what's the slope of the yellow line?

Answer:

12. Look at the following table, which contains polynomial functions and its derivatives.

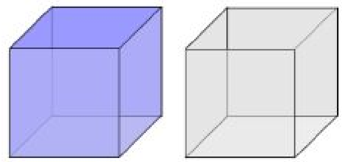
$f(x)$	$f'(x)$
...	...
$3x^5 + 2x^4 + x^3 + x^2 + x + 2$	$15x^4 + 8x^3 + 3x^2 + 2x + 1$
$x^4 + x^3 - x^2 + x - 100$	$4x^3 + 3x^2 - 2x + 1$
$-2x^3 + x^2 + x - \pi$	$-6x^2 + 2x + 1$
$-x^2 - \frac{3}{4}x + \frac{7}{2}$	$-2x - \frac{3}{4}$
$x - \sqrt{2}$	1

Identify pattern(s) in the previous table that helps you to conclude how you should proceed to differentiate $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$, where the $a's$ are real numbers usually called the *coefficients* of the polynomial function. Then, write a formula for $f'(x)$. *Hint:* Recall, from the previous problem, that the derivative of a constant is zero. That's the reason to have *one less* term on the right column (in each case).

Answer:

In summary, consider a polynomial of degree n, which can be defined as a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$, where the $a's$ are real numbers usually called the *coefficients* of the polynomial. Then, its derivative is defined by $f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + ... + a_1$.

13. Look at the following picture.



That is, it is given, from left to right, a filled cube and an empty cube. The *surface area* of the cube can be computed via the following formula: $A = 6x^2$, where x is the length of its edges. On the other hand, the *volume* of the cube can be computed via the following formula: $V = x^3$, where x is the length of its edges.

a. If a cube has a volume of 1 cubic meter. What's its surface area (in square meters)?
   Answer:

b. If a cube has a surface area of 6 square meters. What's its volume (in cubic meters)?
   Answer:

c. What's the relation between A and V'?
   Answer:

Note. The previous example, and the example of the sphere, gives rise to a *very* interesting question: Is there always a relation, via derivatives, between the surface area and the volume of a 3-dimensional object? Work with this question requires both, a lot of thinking, and more knowledge than the one you learned Today. But, there are discovered "derivative relationships" between volume and surface area. In the future, when you obtain more Math knowledge, maybe you will want to take a look at the paper mentioned in the "Sources used" part (by the end of this worksheet).

Final problem set

14. Fill the right column of the following table. That is, differentiate each of the given functions.

$f(x)$	$f'(x)$
$x^5 + x^4 + x^3 + x^2 + x + 1$	
$5x^7 - \pi x^3 + x^2 - 100x + 987654321$	
$(x^2 + x)(x^3 - x)$	
$(-4x^4 - x)(-\sqrt{2}x^6 + x^3 + 1)$	
$\frac{x^6+x^5+x^4+x^3+x^2+x}{x}$	
$-\frac{\pi^2 x^9-987654321x^5-\sqrt{7}x^4+e^2x^7}{\pi e\sqrt{7}x^3}$	
$987654321123456789\,x$	
$7777777777777777777777777777$	
$f(x) + f'(x)$	

15. Fill the right column of the following table. That is, differentiate *twice* each of the given functions.

Note: The left column of the table is the same as for the table of the previous problem.

$f(x)$	$f''(x)$
$x^5 + x^4 + x^3 + x^2 + x + 1$	
$5x^7 - \pi x^3 + x^2 - 100x + 987654321$	
$(x^2 + x)(x^3 - x)$	
$(-4x^4 - x)(-\sqrt{2}x^6 + x^3 + 1)$	
$\frac{x^6+x^5+x^4+x^3+x^2+x}{x}$	
$-\frac{\pi^2 x^9 - 987654321x^5 - \sqrt{7}x^4 + e^2 x^7}{\pi e \sqrt{7} x^3}$	
$987654321123456789x$	
$7777777777777777777777777777$	
$f(x) + f'(x)$	

16. Consider the following polynomial function $f(x) = x^{77711987654321999}$. What are you going to obtain after differentiating 77711987654321999 times?

Answer:

17. Consider the following polynomial function

$f(x) = x^{77711987654321999} + x^{77711987654321998} + x^{77711987654321997}$. What are you going to obtain after differentiating 77711987654321998 times?

Answer:

18. Again, suppose that you want to walk from your house to the house of your best friend. Suppose it is located 5 kilometers from your house, and it takes you a *certain time* (in minutes). Imagine that, at the beginning of your journey, you stay 10 minutes *without walking*, since you're playing nintendo switch. Then, the scary dog suddenly appears, so that you start to run at a constant speed from your house until 2 kilometers, where you escaped from the scary dog. Then, you walk, or run slower, at a constant speed until you achieve 3 kilometers. Here, you start to run at a constant speed, since you see that the terminator in problem 3. starts to walk towards you, until the end of your journey. Thankfully, you noticed that the Terminator is no longer chasing you when arrived to the house of your best friend.

For the following questions, assume the following.

- When playing nintendo switch (first 10 minutes), your traveled distance, or position, is given by $f(x) = 0$, where x is the time and $f(x)$ is the traveled distance.
- When escaping from the scary dog, your traveled distance, or position, is given by $f(x) = \frac{x}{5} - 2$, where x is the time and $f(x)$ is the traveled distance.
- After escaping from the scary dog, your traveled distance, or position, is described by $f(x) = \frac{x}{10}$, where x is the time and $f(x)$ is the traveled distance.
- When escaping from the Terminator, your traveled distance is described by $f(x) = \frac{x}{6} - 2$, where x is the time and $f(x)$ is the traveled distance.

a. At what time you start to walk after escaping from the scary dog?

Answer:

b. At what time you start to run again (that is, when the Terminator starts to walk towards you)?

Answer:

c. How much time takes you to arrive to the house of your friend?

Answer:

d. Determine the slope, or derivative, during the period of time from 0 to 10 minutes.

Answer:

e. Determine the slope, or derivative, during the period of time from 10 to the time you computed in a.?

Answer:

f. Determine the slope, or derivative , during the period of time from the time you computed in a. until the time you computed in b.

Answer:

g. Determine the slope, or derivative, during the period of time from the time you computed in b. until the time you computed in c.

Answer:

h. What was the scariest creature (the scary dog or the Terminator)? Justify your answer according to previous answer(s).

Answer:

i. Determine the intercept of each of the following pair of functions.

- $f(x) = 0$ and $f(x) = \frac{x}{5} - 2$
- $f(x) = \frac{x}{5} - 2$ and $f(x) = \frac{x}{10}$
- $f(x) = \frac{x}{10}$ and $f(x) = \frac{x}{6} - 2$

Answer:

j. Draw a sketch of the graph *distance Vs. time* including all the information/computations of questions a.-i.

Answer:

Sources used/Acknowledgment

1) Thanks to Dr. Thornton and Dr. Johnson for the advising in how to prepare/improve a presentation for the Math Circle.

2) YouTube video (channel: Lighthouse): What is Differentiation - January 23, 2016
3) YouTube video (channel: Derivando): ¿Qué son las derivadas? - April 7, 2016
4) Paper: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.598.3691&rep=rep1&type=pdf
5) Links to pictures (except graphs in pp. 1-7 and 14, since I made them using paint app):
    - https://www.boredpanda.com/scary-dogs-muzzle-werewolf/?utm_source=google&utm_medium=organic&utm_campaign=organic
    - https://getoutside.ordnancesurvey.co.uk/guides/7-things-you-never-expected-in-britains-hills/
    - https://www.cnet.com/reviews/nintendo-switch-review/
    - https://www.pocket-lint.com/games/reviews/nintendo/149640-nintendo-switch-lite-review
    - https://www.theatlantic.com/entertainment/archive/2014/12/have-we-reached-peak-reboot-with-terminator-genisys/383449/
    - Graphs of $\frac{x}{10} + 2$ and $\frac{1}{10}$: WolframAlpha
    - https://betterexplained.com/articles/a-gentle-introduction-to-learning-calculus/ (circle-disk and sphere-ball)
    - https://tex.stackexchange.com/questions/256236/how-to-draw-a-cube-completely-filled-with-water-the-other-with-water-by-half-an (cubes)