# Team Round 11901

1. A triangle has coordinates $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$, and area 5. If the value of each of the 6 coordinates is multiplied by 2, compute the new area of this triangle.

2. Triangle OTP has sides OT = 13, OP = 12, and TP = 5. A point M distinct from P exists such that OM = 12. Find the maximum possible area of triangle TPM.

3. Anna has a large collection of economics books and two very tall bookcases. The first bookcase can hold 39 books on each shelf; when she places all her books on it, one shelf only holds 31 books. The second bookcase can hold 91 books on each shelf; when she places all her books on it, one shelf only holds 83 books. In both cases, Anna always completely fills as many shelves as possible. What is the smallest number of economics books Anna could have?

4. John is planning to paint a room. He needs to paint 4 walls and the ceiling. For both John and his friend Omar, the ceiling takes 3 times as long to paint as a wall does. Halfway through the time required to paint the room by himself, John is joined by Omar. Omar can hold and move the ladder while John paints the ceiling so the ceiling takes exactly as long as a wall, start painting alongside John, or do some combination of the two. If John can paint a single wall in 5 minutes, and Omar can paint a single wall in 6 minutes, how many minutes, in total, will it take them to paint the entire ~~wall~~ room if they choose the fastest option? Express your answer as a common fraction.

5. A right triangle has integer side lengths, and one of the legs has length 99. Find the greatest possible sum of the lengths of the other two sides of the triangle.

6. The Notfake Middle School Math Club is preparing for two different competitions. For the first, the Indisputably Real Local, the club must send a 5 person team, with a designated team captain as one of the members. For the second, the Rivers, Oceans, and Forests League, the club must send a 6 person team, with no designated captain. Students can compete at either, both, or neither competition. The math club's coach finds that the club has more possible team options (including the choice of captain) for the IRL than the ROFL. What is the maximum number of members the math club could have?

7. Blake and Krista are giving away candy bars to the nine children on their street for Halloween. They have ~~up to~~ two dozen identical candy bars to give away, and want each child to receive at least two candy bars, but no child to receive more than one more candy bar than any other child. If Blake and Krista may keep some candy bars for themselves, how many different ways can they give away candy bars to the children?

8. A four-digit integer has the interesting property that it is a multiple of both 3 and 4, and when its hundreds digit is swapped with its units digit, and its tens digit is swapped with its thousands digit, the resulting number is also a multiple of 3 and 4. How many four-digit numbers have this property?

 

9. A person is called pythagorean if the year they are born, the year they die, and their age at death are all perfect squares. Assuming people will never live for more than 150 years, what is the last year in which a pythagorean could be born?

10. Regular hexagon ABCDEF has side length 2 and center P and lies in plane $\mu$. Six spheres of radius 1 are tangent to $\mu$ and lie on the same side of $\mu$, with the vertices of ABCDEF as the points of tangency. A hexagonal pyramid has height 2, has its apex lying on P, and also lies on the same side of $\mu$ as the spheres with its base parallel to $\mu$. Each (triangular) lateral face of the pyramid is tangent to one of the spheres, and the point of tangency lies on the altitude of the face from P. Compute the volume of the pyramid.

 

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1. By multiplying the coordinates by 2, we are stretching the side lengths of the triangle by a factor of 2, and thus multiplying the area by 4, for a new area of 20.

2. Note that M and P fall on a circle of radius 12 centered at O. Thus MP $\leq$ 24 (the length of a diameter of the circle), and therefore the altitude from M to side TP is at most 24. Now, since OP and TP are perpendicular, TP must be tangent to our circle at P. Thus the diameter through P is perpendicular to TP, and our maximum area is $\frac{5 \cdot 24}{2} = 60$.

3. Let the number of books be N. We have $N \equiv -8 \bmod 39$ and $N \equiv -8 \bmod 91$. Thus $N \equiv -8 \bmod 273$, since $273 = \text{lcm}(39, 91)$. The smallest such N is $-8 + 273 = 265$.

4. Since the ceiling takes three times as long as a wall, we can treat it as three walls, in which case the room has a total of 7 walls. Therefore, before Omar arrives, John has painted 3.5 walls. When painting the ceiling, Omar can hold the ladder, in which case John takes 5 minutes to paint the ceiling. Otherwise, the ceiling counts as 3 walls, and John and Omar together paint $\frac{1}{5} + \frac{1}{6} = \frac{11}{30}$ walls per minute. Then, Omar might as well help John paint the last half-wall, which would take $\frac{15}{11}$ minutes. Thus, it would take at least $5 + \frac{15}{11} + \frac{35}{2} = \frac{525}{22}$ minutes to finish painting.

5. Let the other two sides be b and c. By the Pythagorean Theorem, $99^2 + b^2 = c^2$. We rearrange and factor to get $(c - b)(c + b) = 99^2$. Thus we want to minimize $c - b$. The obvious choice is $c - b = 1$ and $c + b = (100 - 1)^2 = 10000 - 200 + 1 = 9801$. We can check that this works by showing that c and b exist: $c + b = 2b + 1 = 9801$, and therefore $b = 4900$, $c = 4901$, so we are done.

6. Let the club have n members. There are $5\binom{n}{5}$ choices for the IRL team, and $\binom{n}{6}$ choices for the ROFL, so we must have $5\binom{n}{5} > \binom{n}{6}$. Rearranging gives $\frac{5 \cdot 6!}{5!} > \frac{(n-5)! \cdot n!}{(n-6)! \cdot n!}$, which simplifies to $30 > n - 5$, or $n < 35$. So the maximum number of members is 34.

7. If every child receives 2 candy bars, that accounts for 18 of the 24 on hand. Some children can have a third candy bar, but since there is no way in which everyone can get a third candy bar, no child can get a fourth candy bar. Blake and Krista can choose up to 6 children to receive a third candy bar. This can be done in $\binom{9}{0} + \binom{9}{1} + \binom{9}{2} + \binom{9}{3} + \binom{9}{4} + \binom{9}{5} + \binom{9}{6} = 466$ ways.

 

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8. Let the four-digit number be ABCD. We know that CD is a multiple of 4. We also know that because CDAB is a multiple of 4, that AB is a multiple of 4. Finally, $A + B + C + D \equiv 0 \pmod 3$, and $A, C \neq 0$. We now partition the set of two-digit integers that are multiples of 4 into three categories, based on their remainders when divided by 3. We have $\{12, 24, 36, 48, 60, 72, 84, 96\}$, which are all multiples of 3, $\{16, 28, 40, 52, 64, 76, 88\}$, which have a remainder of 1 when divided by 3, and $\{20, 32, 44, 56, 68, 80, 92\}$, which have a remainder of 2 when divided by 3. There are three ways to assemble ABCD. We can choose both AB and CD from the first set, which can be done in $8 \cdot 8 = 64$ ways. Or we can choose AB from the second set and CD from the third, which can be done in $7 \cdot 7 = 49$ ways. Or we can choose AB from the third set and CD from the second, which can also be done in 49 ways. Altogether, there are $64 + 49 + 49 = \boxed{162}$ such four-digit numbers.

9. Suppose the year the person is born is $a^2$. If the year they die is after $(a + 1)^2$, that person will have to live for at least $(a + 2)^2 - a^2 - 1 = 4a + 3$ years (not $4a + 4$, because (s)he could die before their birthday in year $(a + 2)^2$). Thus $4a + 3 \leq 150$, so $a \leq 36$. On the other hand, if the person dies in year $(a + 1)^2$, then (s)he lives for either $(a + 1)^2 - a^2 = 2a + 1$ years or $2a$ years if (s)he dies before their birthday. The largest perfect square less than 150 is 144, so we use the latter case. This gives $a = 72$, and the latest possible year of birth is $72^2 = \boxed{5184.}$

10. Let O be the center of the sphere that touches $\mu$ at A. Let PXY be the lateral face of the pyramid tangent to this sphere; also let M be the center of the base of the pyramid and Z be the midpoint of XY. Finally, let K be the point at which PXY is tangent to the sphere,such that K lies on PZ. Since PK and PA are common tangents to the sphere, PK = PA, and it follows that $\triangle POA \cong \triangle POK$. Let $\angle OPA = \alpha$, so OA = 1, PA = 2, and $\tan\alpha = \frac{1}{2}$. Since $\angle ZPM = 90 - 2\alpha$, we have $\tan\angle ZPM = \frac{1}{\tan 2\alpha} = \frac{1-\tan^2\alpha}{2\tan\alpha} = \frac{3}{4}$. Also, PM = 2, so $ZM = \frac{3}{2}$. Since ZM is the apothem (inradius) of the base of the pyramid, the base has side length $\sqrt{3}$ and area $\frac{1}{2} \cdot \frac{3}{2} \cdot 6\sqrt{3} = \frac{9\sqrt{3}}{2}$. Therefore, the volume of the pyramid is $\frac{1}{3} \cdot \frac{9\sqrt{3}}{2} \cdot 2 = \boxed{3\sqrt{3}.}$

 