Team Round 12001

1. Ari goes outside and sees a colony teeming with ants! She gets inspired by this and decides that, for her math project on exponents, she will research how many ants are typically in an ant colony and compare this to other quantities in terms of powers of 10. For the lowest estimate of the number of ants in a colony, she comes up with 250, and for the highest estimate, she comes up with $\sqrt{62,500,000,000}$. Let k be the positive difference between the exponents of the lowest and highest estimates when written as powers of 10. Find k.

2. An acute, isosceles triangle has two side lengths of 20. Compute the sum of the possible lengths of the third side, given that its length is an integer.

3. A triangular number is a positive integer that can be written as a sum of the first n positive integers for some positive integer n. For example, 6 is the 3rd triangular number since $6 = 1 + 2 + 3$, and 15 is the 5th triangular number since $15 = 1 + 2 + 3 + 4 + 5$. Compute the 2019th smallest positive integer which is not a triangular number.

4. Compute the sum of all two-digit positive integers whose tens digit is strictly greater than their units digit.

5. Compute the largest prime factor of $4040 \cdot 1009 + 673 \cdot 6060 + 4042 \cdot 1010 + 2020 \cdot 2022$.

6. In the expression $\log(xy) + \log(x^3y^2) + \log(x^6y^3) + \log(x^{10}y^4) + \cdots + \log(x^{210}y^{20})$, the exponents of the y portion of each successive term form an arithmetic sequence, while the exponents of the x portion of each successive term are successive triangular numbers. (Triangular numbers are the sum of the integers from 1 through n, inclusive.) The given expression is equivalent to $c \cdot \log\left(x^ay^b\right)$, where a, b, and c are positive integers and c is as large as possible. Compute $a + b + c$.

7. Call a positive integer *substantial* if the mean of its digits is at least 7. How many positive integers less than 2020 are substantial?

8. Regular hexagon $ABCDEF$ has area A_1. Points G, H, and I are the midpoints of AB, CD, and EF, respectively. The largest circle that can be inscribed in $\triangle GHI$ has area A_2. If $\frac{A_1}{A_2} = \frac{\sqrt{N}}{\pi}$, where N is a positive rational number, compute N.

9. In the coordinate plane, a *lattice point* is a point with integer coordinates, and a *lattice square* is a unit square (square with side length 1) whose vertices are lattice points. The region enclosed by the circle $x^2 + y^2 = 903$ contains 2837 lattice points. Compute the number of lattice squares which lie completely inside this region.

10. On the bonus round of a game show, contestants spin a wheel with 20 panels (each equally likely to be landed on), ranging from \$0.05 to \$1 in 5-cent increments. Contestants have up to two chances to get as close to \$1.00 as possible without going over. If their total dollar amount remains at or below \$1.00 within two spins, they win that dollar amount. If they go over \$1.00, however, they win nothing. If they win \$1.00 on the first spin, they win \$1.00 and do not spin a second time. (The contestant must spin twice, the only exception to this being if they spin \$1.00 on the first spin.) Compute the expected winnings, in cents, from spinning the wheel. Express your answer as a common fraction.

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Relay Round 12001

(1-1) Let $T = 76$. Let N be the units digit of T. The probability of rolling a sum of $N+2$ with a pair of standard dice can be expressed as $\frac{a}{b}$ in lowest terms. Let $K = a+b$. Find the remainder when $K+35$ is divided by 100.

(1-2) Let $T = TNYWR$. Let K be the number of ordered triples of positive integers (x, y, z) that satisfy the equation $x+y+z = T$. Find the remainder when $K+95$ is divided by 100.

(1-3) Let $T = TNYWR$. Let $K = T$. Find the remainder when $K+18$ is divided by 100.

Relay Round 12001

(2-1) Let $T = 46$. Let $\frac{K\sqrt{3}}{16}$ be the area of an equilateral triangle with side length T. Find the remainder when $K + 48$ is divided by 100.

(2-2) Let $T = TNYWR$. When the coefficients of the polynomial $(x+1)^T$ are added, the units digit is K. Find the remainder when $K + 39$ is divided by 100.

(2-3) Let $T = TNYWR$. In a group of T people where everyone shakes everyone else's hand once, there are K handshakes. Find the remainder when $K + 58$ is divided by 100.

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Relay Round 12001

(5-1) Let $T = 51$. Let K be the number of sequences consisting of only 1's and 2's whose terms sum to $\lfloor \sqrt{T+1} \rfloor$ (note that $\lfloor x \rfloor$ represents the greatest integer function). Find the remainder when $K + 96$ is divided by 100.

(5-2) Let $T = TNYWR$. Some list of positive integers sums to $T + 5$. The maximum possible product of such a list of integers can be represented as $2^a 3^b$ for some integers a and b. Let $K = a + b$. Find the remainder when $K + 0$ is divided by 100.

(5-3) Let $T = TNYWR$. Let K be the smallest positive integer such that there are no perfect squares in the interval $[K, K + T]$. Find the remainder when $K + 66$ is divided by 100.

Relay Round 12001

(1-1)	76	(K = 41)	(1-2)	70	(K = 2775)	(1-3)	88	(K = 70)
(2-1)	12	(K = 8464)	(2-2)	45	(K = 6)	(2-3)	48	(K = 990)
~~(3-1)~~	~~58~~	~~(K = 72)~~	~~(3-2)~~	~~36~~	~~(K = 39711)~~	~~(3-3)~~	~~95~~	~~(K = 107)~~
~~(4-1)~~	~~86~~	~~(K = 87)~~	~~(4-2)~~	~~44~~	~~(K = 188131871)~~	~~(4-3)~~	~~31~~	~~(K = 28)~~
(5-1)	17	(K = 21)	(5-2)	8	(K = 8)	(5-3)	92	(K = 26)

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Solutions 12001

Team Round Solutions

1. Observe that $\sqrt{62,500,000,000} = \sqrt{625 \cdot 10^8} = 25 \cdot 10^4$. Let $L = 250$ and $H = 25 \cdot 10^4$, or $250 \cdot 10^3$. We have $\frac{H}{L} = 10^3$ and $k = 3$.

2. By the triangle inequality, the third side length must lie between 1 and 39, inclusive. However, in an acute triangle, where a, b, and c are its side lengths from shortest to longest, $a^2 + b^2 > c^2$, or $c^2 < 800$. Note that $28^2 < 800 < 29^2$, so $1 \leq c \leq 28$. The sum of all integers from 1 to 28, inclusive, is $\frac{28 \cdot 29}{2} = 14 \cdot 29 = 406$.

3. Note that $2080 = 1 + 2 + 3 + \cdots + 64$, so the $2080 - 64 = 2016$th element in the set is 2079. Hence, the 2019th element will be 2083.

4. We want to compute the sum of all integers of the form $\underline{AB}$ where $A > B$. The tens digit portion of this sum is $10 \cdot ((1 \cdot 1) + (2 \cdot 2) + (3 \cdot 3) + \cdots + (9 \cdot 9)) = 2850$. The units digit portion of this sum is $0 + (0 + 1) + (0 + 1 + 2) + \cdots + (0 + 1 + 2 + \cdots + 8)$, or $1 + 3 + 6 + \cdots + 36 = 120$. Thus, the desired total is $2850 + 120 = 2970$.

5. After rewriting the expression as $2020 \cdot 2018 + 2019 \cdot 2020 + 2021 \cdot 2020 + 2020 \cdot 2022$, we have $2020 \cdot 4040 + 4040 \cdot 2020 = 4040^2$. We factor 4040 as $101 \cdot 5 \cdot 2^2$, so the largest prime factor of the expression is 101.

6. By the logarithm addition rule, the LHS simplifies to $\log((x \cdot x^3 \cdot x^6 \cdot \cdots \cdot x^{210} \cdot (y \cdot y^2 \cdot y^3 \cdot \cdots \cdot y^{20})) = \log(x^{\binom{22}{3}} y^{210})$, or $\log(x^{1540} y^{210})$. As $\gcd(1540, 210) = 70$, $c = 70$, and consequently, $a = 22$ and $b = 3$. We have $a + b + c = 95$.

7. We will do casework on the number of digits. The only substantial 4-digit number less than 2020 is 1999, since a number of the form $1xyz$ must have $x + y + z \geq 27$, which forces $x = y = z = 9$ (as the sum of digits must be at least $7 \cdot 4 = 28$). Similarly, $2xyz$ does not work, as $x + y + z \geq 26$, and this cannot be satisfied by an integer from 2000 to 2019. For 3-digit integers xyz, we require $x + y + z \geq 21$, which can be satisfied in $1 + 3 + 6 + 10 + \cdots + 28 = 84$ ways by varying x from 3 to 9. For 2-digit integers xy, $x + y \geq 14$; this condition is satisfied by 59, 68, 69, 77, 78, 79, 86 through 89, and 95 through 99, for a total of 15 2-digit integers. Finally, for single-digit integers, we have 7 through 9. Altogether, the total number of substantial positive integers less than 2020 is $1 + 84 + 15 + 3 = 103$.

8. Without loss of generality, assume the hexagon has side length 2; this will simplify our calculations. (Note that the answer does not change if we use a variable for the side length of the hexagon; this is left as an exercise.) Triangle GHI is equilateral, and its side length, such as $\overline{HI}$, is just 3, as H and I are the midpoints of CD and EF, respectively. Hence, using the area formula $A = rs$, where A, r, and s are the area, inradius, and semiperimeter of $\triangle GHI$, we obtain $r = \frac{\sqrt{3}}{2}$ and $A_2 = \pi r^2 = \frac{3\pi}{4}$. We also have $A_1 = 6 \cdot \frac{2^2\sqrt{3}}{4} = 6\sqrt{3}$, because the hexagon is made up of six equilateral triangles with side length 2. Finally, $\frac{A_1}{A_2} = \frac{8\sqrt{3}}{\pi} = \frac{\sqrt{192}}{\pi}$, so $N =$ 192.

9. Consider the map from lattice squares to lattice points with non-zero coordinates, where each square is mapped to whichever of its vertices is farthest from the origin. Since the vertex is the farthest point from the origin, the vertex is inside the circle if and only if the whole square is inside the circle. Therefore, we just need to count the number of lattice points with non-zero coordinates inside the circle. We are given that there are 2837 lattice points inside the circle, and the points with at least one coordinate which is zero are $(n, 0)$, $(0, n)$ for $-30 \leq n \leq 30$, $n \neq 0$ and $(0, 0)$, giving us a tally of 121 lattice points. Therefore the number of lattice points with non-zero coordinates is $2837 - 121 =$ 2716, which is our answer.

10. Suppose that the contestant does *not* spin \$1.00 on his/her first spin. Let the first spin amount be c cents, and the second spin amount be d cents. If $c = 5$, then $5 \leq d \leq 95$, and the winnings are $c + d$. Otherwise, if $d = 100$, then the winnings are zero. The sum of possible winnings, in cents, is $10 + 15 + 20 + \cdots + 100 = 5(2 + 3 + 4 + \cdots + 20) = 5 \cdot 209$. If $c = 10$, similarly, the winnings sum to $5(3 + 4 + 5 + \cdots + 20) = 5 \cdot 207$. This continues until $c = 95$, at which point the only possible winning is $100 = 5 \cdot 20$. For $c = 100$, the winning is guaranteed to be 100 cents, so we multiply 100 by 20 (to account for the second wheel which is never spun, but could just as well be anything). Thus, the sum of possible winnings is $5(209 + 207 + 204 + 200 + \cdots + 57 + 39 + 20) + 100 \cdot 20 = 5((210 - 1) + (210 - 3) + (210 - 6) + (210 - 10) + \cdots + (210 - (1 + 2 + 3 + \cdots + 19))) + 2000 = 5(210 \cdot 19 - (1 + 3 + 6 + 10 + \cdots + 190)) + 2000 = 5\left(3990 - \binom{21}{3}\right) + 2000 = 5(3990 - 1330) + 2000 = 5(2660) + 2000 = 15,300$. In total, there are $20^2 = 400$ possible outcomes, including the guaranteed winnings for \$1.00 as the first spin. Hence, the expected winnings, in cents, are $\frac{15,300}{400} = \frac{153}{4}$.

