Sprint Answers
12001

Name ____________________

Grade ____________________

School ____________________

Select E only if you cannot determine a uniquely correct answer between A, B, C, and D.

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Scorer Initials ____ Scorer Initials ____

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Sprint Round 12001

1. At the Bay Area Beverage Parlor, 2 boba teas and a milkshake cost \$5.40, while 3 boba teas and 4 milkshakes cost \$15.10. What is the cost of a boba tea, in cents?

   (A) 140 (B) 135 (C) 130 (D) 125 (E) Other

2. The total bill at a restaurant is \$58 before sales tax. Sales tax is 4%. If leaving a tip of 17.5%, how many cents would one save by tipping on the total bill not including the sales tax as opposed to tipping on the total bill including the sales tax? Round your answer to the nearest whole number.

   (A) 42 (B) 40 (C) 39 (D) 38 (E) Other

3. From a 3 by 3 square grid of white squares, three of the nine squares are randomly chosen and shaded. Compute the probability that at least one row or column has two or more shaded squares.

   (A) $\frac{6}{7}$ (B) $\frac{13}{14}$ (C) $\frac{11}{14}$ (D) $\frac{20}{21}$ (E) Other

4. Thorin's Company includes 13 elves, a Hobbit, and Gandalf the Wizard. They sit in two rows for a picture, with eight creatures in the back row and seven in the front row. The Hobbit must be in the front row, and Gandalf must be in the back row. Assuming that the elves are indistinguishable, how many seating arrangements are there?

   (A) 13 (B) 42 (C) 14 (D) 56 (E) Other

5. Robert the builder (well, really 4-year-old construction-worker-hopeful) has an ample supply of 3-inch-tall, 4-inch-tall, and 5-inch-tall blocks in his playset. He wants to use any number of blocks to build a structure that is exactly 14 inches tall. In how many distinguishable ways can he do this?

   (A) 14 (B) 12 (C) 16 (D) 10 (E) Other

6. What is the root of the equation $2^{x+2} - 2^x = 6$?

   (A) 1 (B) 2 (C) 4 (D) -1 (E) Other

7. A pan of cornbread is 20 inches long by 19 inches wide. At most how many 1.5×1.5 inch cornbread squares can be cut from the pan?

   (A) 156 (B) 160 (C) 144 (D) 169 (E) Other

8. Consider the four vertices of a square in the plane. How many right triangles with a 45° angle have at least two vertices that are also vertices of the square?

   (A) 28 (B) 24 (C) 10 (D) 20 (E) Other

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9. In a multi-purpose skyscraper, there is a lobby floor and some number of residential and business floors. Every residential floor houses exactly 5 tenants, and every business floor employs exactly 3 workers. If the total number of tenants equals the total number of workers, and the number of floors in the skyscraper is $N > 1$, compute the sum of the ten smallest possible values of N.

   (A) 480 (B) 405 (C) 450 (D) 490 (E) Other

10. Compute $\sqrt[6]{4+2\sqrt{3}} \cdot \sqrt[3]{\sqrt{3}-1} \cdot \sqrt[3]{4}$.

    (A) 1 (B) 5 (C) 4 (D) 3 (E) Other

11. A square pyramid has base length 12. A square cuts through the pyramid parallel to the base at 5 units above the base, forming a square cross-section with area 81. Compute the volume of the original pyramid.

    (A) 1200 (B) 1440 (C) 320 (D) 960 (E) Other

12. The third term of a finite geometric series with 10 terms is 4, and the first term is 1 smaller than the second term. If the sum of the terms in the series is s, compute s.

    (A) 1024 (B) 511 (C) 512 (D) 1023 (E) Other

13. Each term of a ten-term sequence is either a 1 or a 0. How many such ten-term sequences do not have any three consecutive terms of "1, 1, 1"; "0, 0, 0"; "1, 0, 1"; or "0, 1, 0"?

    (A) 1 (B) 4 (C) 16 (D) 2 (E) Other

14. For each positive integer x, let $f(x)$ be the integer closest to $\sqrt{x}$. What is the sum of the values of $f(f(n))$, evaluated for all integers n from 1 to 100, inclusive?

    (A) 246 (B) 202 (C) 252 (D) 352 (E) Other

15. Square $ABCD$ has side length 5. Points E, F, G, and H lie on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ respectively, such that $AE = 1$, $BF = 2$, $CG = 3$, and $DH = 4$. Compute the area of quadrilateral $EFGH$.

    (A) $\frac{27}{2}$ (B) 10 (C) 12 (D) 15 (E) Other

16. The quarternary (base 4) number 120303213_4 is written in octal (base 8). What is the base 10 sum of the digits of this octal number?

    (A) 23 (B) 17 (C) 19 (D) 20 (E) Other

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17. Triangle ABC has $AB = 7$, $BC = 8$, $CA = 9$. The angle bisector of $\angle A$ and the angle bisector of $\angle B$ intersect $\overline{BC}$ and $\overline{AC}$ at points D and E, respectively. Compute the area of $\triangle EDC$.

(A) $\frac{12\sqrt{5}}{5}$ (B) $6\sqrt{5}$ (C) $\frac{24\sqrt{5}}{5}$ (D) $\frac{36\sqrt{5}}{5}$ (E) Other

18. If $0 < \theta < 2\pi$ is a real number such that $\sin(\theta)\tan(\theta) = \frac{1}{4}$, compute $\cos(\theta)$.

(A) $\frac{8-\sqrt{65}}{8}$ (B) $\frac{\sqrt{65}-1}{8}$ (C) $\frac{\sqrt{17}-4}{8}$ (D) $\sqrt{5}-2$ (E) Other

19. One night, Jade falls asleep at a random time between 11 PM and 12 AM. She has set her alarm clock for 7 AM, but her alarm is malfunctioning, so it goes off at a random time between 6 AM and 8 AM. What is the probability that Jade gets at least 7 hours and 45 minutes of sleep?

(A) $\frac{1}{32}$ (B) $\frac{3}{16}$ (C) $\frac{1}{4}$ (D) $\frac{1}{8}$ (E) Other

20. In regular hexagon $ABCDEF$, an ant starts at vertex A and travels along the diagonals of the hexagon to reach other vertices. It can traverse each diagonal at most once. In how many ways can the ant visit each other vertex exactly once and then return to point A?

(A) 3 (B) 12 (C) 8 (D) 6 (E) Other

21. Cameron and Julia are running laps around a circular track, starting from the same point. Initially, they run in opposite directions at constant speeds, with Cameron taking 3 minutes to complete a lap, and Julia taking 2 minutes to complete a lap. Each time Cameron and Julia meet, Julia switches directions instantly while maintaining the same pace. After how many minutes will both runners meet back at their starting point for the first time?

(A) 54 (B) 60 (C) 90 (D) 36 (E) Other

22. Equilateral triangle ABC has side length 1. Point P lies in the plane of $\triangle ABC$ such that $\angle CBP$ is right and $\angle BCP = 30°$. Compute the largest possible length of line segment $\overline{AP}$.

(A) $\sqrt{7}$ (B) $\frac{\sqrt{21}}{3}$ (C) $\sqrt{3}$ (D) $\frac{7}{3}$ (E) Other

23. The expression $A = (1)(1-2)(1-2+3)(1-2+3-4)\cdots(1-2+3-4+\cdots+2019-2020)$ is the product of 2020 smaller expressions which end with the integers from 1 to 2020. (Each smaller expression ending with k lists the integers from 1 to k in increasing order, and alternates between subtracting and adding those terms, starting with the subtraction of the second integer.) Compute the number of zeros at the end of the base-10 expression of A.

(A) 250 (B) 500 (C) 502 (D) 501 (E) Other

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24. Vivek is studying for a final exam. Every day that he studies, he gains 100 intel at the beginning of the day, starting from zero intel. However, each day, he loses $\frac{1}{5}$ of his existing intel at the end of the day, whether he studies or not. If Vivek studies for 60 days before exam day but not on exam day, to the nearest whole number, how much intel does Vivek have at the beginning of exam day? It may be useful to know that $4^{60} \cdot 800 < 5^{60}$.

(A) 500 (B) 499 (C) 400 (D) 399 (E) Other

25. Compute the distance between the intersection points of the graphs of $(x-16)^2+y^2=256$ and $(x-8)^2+y^2=144$.

(A) $3\sqrt{15}$ (B) 10 (C) $6\sqrt{15}$ (D) $20\sqrt{7}$ (E) Other

26. If x is a real number for which $\sin(x) = \tan(x) - \cos(x)$, then $\sin(2x) = P(\sin(x))$, where P is a quadratic polynomial with integer coefficients. Compute $P(10)$.

(A) 222 (B) 118 (C) 122 (D) 182 (E) Other

27. Say a positive integer is a *burger* if its first and last digits are the same, but it is not a palindrome. Let S be the sum of all four-digit burger numbers. Compute the remainder when S is divided by 1000.

(A) 320 (B) 640 (C) 480 (D) 0 (E) Other

28. Let x be a positive integer such that $\log_2(\log_{16} x) = M$ and $\log_{16}(\log_2 x) = N$. If $M = 2N$, compute the last two digits of x (in base 10).

(A) 56 (B) 36 (C) 16 (D) 76 (E) Other

29. If x is a real number such that $0 \le x < \frac{\pi}{2}$ and $\sin^4(x)+\cos^4(x) = \frac{5}{9}$, compute the sum of all possible values of $\tan(x)$.

(A) $2\sqrt{2}$ (B) $\frac{\sqrt{2}}{2}$ (C) $\frac{3\sqrt{2}}{2}$ (D) $\frac{5\sqrt{2}}{2}$ (E) Other

30. Kotori has eight acquaintances. Each of these acquaintances may or may not be good friends with Kotori. She wants to draw a graph representing her acquaintances' ties with each other and herself, with a line connecting two people if and only if they are good friends. Every one of Kotori's acquaintances is good friends with exactly one of her other acquaintances. If N is the number of different graphs she could draw under this condition, find the remainder when N is divided by 1000.

(A) 160 (B) 80 (C) 480 (D) 880 (E) Other

Answer Key 12001

Sprint Round Answers

1. C
2. E
3. B
4. D
5. E
6. A
7. A
8. A
9. C
10. E
11. D
12. D
13. B
14. C
15. C
16. A
17. E
18. B
19. E
20. D
21. E
22. B
23. C
24. C
25. C
26. E
27. D
28. B
29. C
30. D

Target Round Answers

1. 675
2. $\frac{\sqrt{3}}{2}$
3. 71
4. L
5. $\frac{3+\sqrt{3}}{6}$
6. $\frac{4+8\sqrt{3}}{11}$
7. 11
8. 25

Team Round Answers

1. 3
2. 406
3. 2083
4. 2970
5. 101
6. 95
7. 103
8. 192
9. 2716
10. $\frac{153}{4}$

Solutions
12001

Sprint Round Solutions

1. Let the cost of a boba tea be b and the cost of a milkshake m. Then we have $2b + m = 5.40$ and $3b + 4m = 15.10$. Use any method you like to solve this system; you should end up with $b = 1.30$ and $m = 2.80$. Therefore, the cost of a boba tea is 130 cents.

2. If one tips on the tax, the extra amount is $\$58 \cdot 4\% \cdot 17.5\% = \$58 \cdot 0.7\%$, or $\$\frac{58}{100} \cdot \frac{7}{10} \approx \$0.41$.

3. In total, there are $\binom{9}{3} = 84$ ways to shade three distinct squares from the nine squares in the grid. Of these, $3! = 6$ have all three squares in distinct rows and columns, so the probability that this does *not* occur is $\frac{84-6}{84} = \frac{13}{14}$.

4. There are 8 spots for Gandalf. There are 7 spots for the Hobbit, regardless of where Gandalf sits. The elves must go in the remaining spots, and since they all look alike, it doesnt matter how they are seated. There are 56 distinguishable seating arrangements.

5. Robert can use the following combinations: 3-3-4-4, 3-3-3-5, or 4-5-5. (With 5 or more blocks, the height is guaranteed to be at least 15 inches, and with two or fewer blocks, 14 inches is clearly not attainable.) The number of distinguishable towers is then $\frac{4!}{2!^2} + \frac{4!}{3!} + \frac{3!}{2!}$, which is $6 + 4 + 3 = 13$.

6. Simplifying the left side, $2^{x+2} - 2^x = 2^2 \cdot 2^x - 2^x$, which is $3 \cdot 2^x$. Therefore $3 \cdot 2^x = 6$, and $2^x = 2$, which means that $x = 1$.

7. We can cut at most $\lfloor \frac{20}{1.5} \rfloor \cdot \lfloor \frac{19}{1.5} \rfloor = 13 \cdot 12 = 156$ cornbread squares.

8. We split the problem into cases. If the triangle shares just two vertices which constitute an edge of the square, there are 4 ways for each edge (two where the square edge is a leg of the triangle, and two where it is the triangle's hypotenuse), resulting in $4 \cdot 4 = 16$ triangles for this case. If the triangle shares just two vertices which constitute a diagonal of the square, there are 4 ways for each diagonal (all of which use the diagonal as a leg of the triangle), so there are 8 triangles for this case. If the triangle shares all three vertices with the square, we can count 4 triangles. In all, there are $16 + 8 + 4 = 28$ triangles.

9. Note that for every three residential floors, there must be five business floors. Thus, all positive integer numbers of floors that are 1 more than a multiple of 8 (excluding 1 itself) can be the floor count of the skyscraper. We seek the sum $9 + 17 + 25 + \cdots + 81$, which is $5(9 + 81) = 450$.

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10. Observe that $4+2\sqrt{3} = (1+\sqrt{3})^2$ (if this is not clear, try setting $\sqrt{4+2\sqrt{3}} = a+b\sqrt{3}$ arbitrarily at first, then squaring both sides and equating coefficients), $\sqrt[6]{4+2\sqrt{3}} = \sqrt[6]{(\sqrt{3}+1)^2}$. Then $\sqrt[6]{4+2\sqrt{3}} \cdot \sqrt[3]{\sqrt{3}-1} = \sqrt[6]{(1+\sqrt{3})^2} \cdot \sqrt[6]{(\sqrt{3}-1)^2}$, which by the difference of squares is $\sqrt[6]{2^2} = \sqrt[6]{4}$. Finally, multiplying by $\sqrt[3]{4} = \sqrt[6]{4^2}$ yields $\sqrt[6]{4^3} = \sqrt[6]{2^6} = 2$.

11. The side length of the square in the cross-section is $\sqrt{81} = 9$, so it lies $\frac{9}{12} = \frac{3}{4}$ of the way from the apex of the pyramid to the base. Thus, the pyramid's height is $\frac{5}{1-\frac{3}{4}} = 20$, and its volume is $\frac{1}{3} \cdot 12^2 \cdot 20 = 960$.

12. Let the common ratio of the series be $r = \frac{1}{a}$, so that the first term is $4a^2$ and the second term is $4a$. Then $4a^2 = 4a - 1$, and $4a^2 - 4a + 1 = 0$. Then $(2a-1)^2 = 0$ and $a = \frac{1}{2}$, so $r = 2$. By the finite geometric series formula, the sum of the terms in the series $1+2+4+8+\cdots+512$ is $\frac{2^{10}-1}{2-1} = 1023$.

13. The first two digits in the string can be either 0 or 1, for $2^2 = 4$ choices thus far. But each subsequent digit is restricted to 1 possibility, since the constraints eliminate one of the two possibilities for each set of two digits that precedes it. Hence, there are just 4 strings that satisfy the condition (regardless of length).

14. Note that, for $k \leq 100$, $f(k) \leq 10$. In addition, when $k = 1$ or $k = 2$, $f(k) = 1$ (as $1.5^2 = 2.25 > 2$); when $k = 3, 4, 5, 6$, $f(k) = 2$, as $2.5^2 = 6.25 > 6$, and when $k = 7, 8, 9, 10$, $f(k) = 3$. Hence, we need to find the number of times $f(f(k)) = 1$, $f(f(k)) = 2$, and $f(f(k)) = 3$. We have that $f(f(k)) = 1$ whenever $f(k) = 1, 2$, or $k \leq 6$. For $7 \leq k < 6.5^2 = 42.25$, $3 \leq f(k) \leq 6$ and $f(f(k)) = 2$. Finally, for $k \geq 43$, $7 \leq f(k) \leq 10$ and $f(f(k)) = 3$. Hence, the desired sum is equal to $1 \cdot 6 + 2 \cdot (42 - 7 + 1) + 3 \cdot (100 - 43 + 1) = 6 + 72 + 174 = 252$.

15. The area of $EFGH$ is the area of $ABCD$, 25, minus the combined areas of $\triangle HAE$, $\triangle EBF$, $\triangle FCG$, and $\triangle GDH$. Respectively, these are $\frac{1^2}{2} = \frac{1}{2}$, $\frac{4 \cdot 2}{2} = 4$, $\frac{3^2}{2} = \frac{9}{2}$, and $\frac{2 \cdot 4}{2} = 4$. Summing these and subtracting from 25 yields 12 as the area of $EFGH$.

16. We can write $N = 4^8 + 2 \cdot 4^7 + 3 \cdot 4^5 + 3 \cdot 4^3 + 2 \cdot 4^2 + 1 \cdot 4 + 3 \cdot 4^0 = 2 \cdot 8^5 + 8^5 + 6 \cdot 8^3 + 3 \cdot 8^2 + 4 \cdot 8 + 7 = 3 \cdot 8^5 + 6 \cdot 8^3 + 3 \cdot 8^2 + 4 \cdot 8 + 7$. Thus, the base-8 representation of N is 306347, which has a digit sum of 23.

17. First, draw the A-altitude down to point P, which creates two right triangles $\triangle APB$ and $\triangle APC$, with $AP = 3\sqrt{5}$ (using Heron's formula), $BP = 2$, and $PC = 6$. Applying the Angle Bisector Theorem, we obtain $\frac{8}{15} \cdot 3\sqrt{5} = \frac{8\sqrt{5}}{5}$ as the vertical distance from E to $\overline{BC}$. Furthermore, we have $DC = \frac{9}{16} \cdot 8 = \frac{9}{2}$, so the area of $\triangle EDC$ is $\frac{1}{2} \cdot \frac{8\sqrt{5}}{5} \cdot \frac{9}{2} = \frac{18\sqrt{5}}{5}$.

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18. Since $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, we have $4\sin^2(\theta) = \cos(\theta)$. As $\sin^2(\theta) + \cos^2(\theta) = 1$, it follows that $4 - 4\cos^2(\theta) = \cos(\theta)$, or $4\cos^2(\theta) + \cos(\theta) - 4 = 0$, or $\cos(\theta) = \frac{-1\pm\sqrt{65}}{8}$. Since $-1 \leq \cos(\theta) \leq 1$, $\cos(\theta) = \frac{\sqrt{65}-1}{8}$.

19. Jade can fall asleep any time between 11 PM and 12 AM; she can wake up at any time between 6 AM and 8 AM. If we plot her sleeping time on the x-axis of a graph, and her waking time on the y-axis, we use the restrictions $0 \leq x \leq 1$ and $7 \leq y \leq 9$ by setting 11 PM to our intial time, and compute the rest of the times relative to that. We then end up with a geometric probability problem where we must find the area of the region above the line $y - x = 7.75 = \frac{31}{4}$. This region is a trapezoid with base lengths $\frac{5}{4}$ and $\frac{1}{4}$, and height 1, hence an area of $\frac{3}{4}$. The total area of all possible outcomes on the graph is $1 \cdot 2 = 2$, for a probability of $\frac{\frac{3}{4}}{2} = \frac{3}{8}$.

20. The ant has 3 choices for the vertex it can travel to next (not A itself, nor the two vertices adjacent to it). It then has 2 choices for the next vertex, and then only 1 choice for the third vertex it travels to, since the other paths are edges of the hexagon. Finally, all other movements are forced. This yields a total of $3! = 6$ ways.

21. Cameron runs at a speed which is $\frac{2}{3}$ of Julia's speed. Say the circle has circumference 5, with 5 equally-spaced markers (including 0), labeled 0-4, clockwise around the circle, and with the starting point located at the 0 mark. Suppose Cameron begins by running clockwise and Julia by running counter-clockwise. The two runners first meet at point 2, at which point Cameron and Julia beginning running in the same direction. From there, for every 3 points Julia passes, Cameron will only pass 2 points, so the distance between them increases by 1 point every 3 points Julia passes. Thus, once the two meet again, they will be at point 2 (Cameron having run 2 laps, or passing 10 marked points), after which Julia promptly runs counter-clockwise again and the process repeats. This repeats 4 more times ($0, 2, 4, 1, 3$ being their meeting points before 0), alternating between Cameron and Julia running in opposite and similar directions. As a result, Cameron and Julia run in opposite directions 5 times and in similar directions 4 times, so the number of points crossed by Cameron is $5 \cdot 2 + 4(5 \cdot 2) = 50$, which corresponds to 10 laps, or 30 minutes. (Indeed, we can verify that Julia runs 15 laps, or 75 points.)

22. Since $\angle CBP$ is right, $\overline{BP} \perp \overline{CB}$, and $\triangle CBP$ is a 30-60-90 triangle. Then $CP = \frac{2\sqrt{3}}{3}$. Notice also that $\triangle ACP$ is a right triangle, with $AC = 1$, so $AP = \sqrt{AC^2 + CP^2} = \sqrt{1^2 + \left(\frac{2\sqrt{3}}{3}\right)^2} = \frac{\sqrt{21}}{3}$.

23. Let S_n be the value of the nth expression. We observe that $S_n = \frac{n+1}{2}$ if n is odd, and $S_n = -\frac{n}{2}$ if n is even. Thus, $\prod_{k=1}^{2020} S_k = \prod_{i=1}^{1010} \frac{(2i-1)+1}{2} \cdot \prod_{j=1}^{1010} \left(-\frac{2j}{2}\right) = 1010! \cdot 1010!$, or $(1010!)^2$. Using Legendre's formula, the number of trailing zeros of $1010!$ is $\lfloor \frac{1010}{5} \rfloor + \lfloor \frac{1010}{5^2} \rfloor + \cdots = 202 + 40 + 8 + 1 = 251$, so the number of trailing zeros of $(1010!)^2$ is $251 \cdot 2 = 502$.

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24. If the amount of intel that Vivek gains each day equals the amount he loses each day, his intel level will remain constant. Hence let I be Vivek's final intel level, or the value that it approaches as the time passed approaches infinity. Then $\frac{4}{5}(I+100) = I$, and solving yields $I = 400$. In particular, after 60 days, his intel will be equal to $400 \cdot \left(1 - \frac{4^{60}}{5^{60}}\right)$. Applying the given hint, ($\frac{4^{60}}{5^{60}} < \frac{1}{800}$), this still rounds to 400.

25. Subtracting the second equation from the first gives us $(x-16)^2 - (x-8)^2 = 112$. By the difference of squares, $((x-16)+(x-8))((x-16)-(x-8)) = 112$, or $(2x-24)(-8) = 112$, and $x = 5$. Substituting this value in for x yields $y = \pm 3\sqrt{15}$, so the distance is $6\sqrt{15}$.

26. We have $\sin(x) = \frac{\sin(x)}{\cos(x)} - \cos(x)$, or $\sin(x)\cos(x) = \sin(x) - \cos^2(x)$ after multiplying through by $\cos(x)$. Rewriting the left-hand side as $\frac{1}{2}\sin(2x)$, it follows that $\sin(2x) = 2\sin(x) - 2\cos^2(x) = 2\sin(x) - 2(1-\sin^2(x)) = 2\sin^2(x) + 2\sin(x) - 2$. Hence $P(x) = 2x^2 + 2x - 2$, giving $P(10) = 218$.

27. If the first and last digits of a four-digit number are both d ($1 \le d \le 9$), then it is a burger, unless its two middle digits are both the same, in which case it is a palindrome. Thus, we want to compute the sum of all four-digit numbers of the form $\overline{dabd}$, minus the sum of all four-digit numbers of the form $\overline{daad}$, where a and b are digits. If $d = 1$, then we want to compute $1001 + 1011 + 1021 + \cdots + 1991$, which evaluates to $50 \cdot 2992 = 149,600$. For all $d \ge 2$, we can just add $1001 \cdot 100(d-1) = 100,100(d-1)$ to the total for $d = 1$, as we are adding $\overline{[d-1]00[d-1]} = 1001(d-1)$ to each individual burger number in the total (where $[d-1]$ is the digit for the expression $d-1$). This yields a total of $149,600 + (149,600 + 100,100 \cdot 1) + (149,600 + 100,100 \cdot 2) + \cdots + (149,600 + 100,100 \cdot 8) = 149,600 \cdot 9 + 100,100(1+2+3+\cdots 8) \equiv 400 + 100,100 \cdot 36 \equiv 400 + 600 \equiv 0 \bmod 1000$. However, we must then subtract the palindromes, namely $1001, 1111, 1221, \cdots, 1991$ for $d = 1$ (and similarly for $d = 2, 3, 4, \cdots 9$). This yields the sum $(1001 + 1111 + 1221 + \cdots + 1991) + (2002 + 2112 + 2222 + \cdots + 2992) + \cdots + (9009 + 9119 + 9229 + \cdots + 9999) = 5(2992 + 4994 + \cdots + 19008) = 5(9 \cdot 11000) = 495,000 \equiv 0 \bmod 1000$, so the sum of all four-digit burgers is 0 mod 1000.

28. Let $x = 2^{2^a} = 16^{\frac{2^a}{4}} = 16^{2^{a-2}}$; then $\log_{16} x = 2^{a-2}$ and $\log_2(\log_{16} x) = a - 2 = M$. Similarly, $\log_2 x = 2^a$ and $\log_{16} 2^a = \log_{16} 16^{\frac{a}{4}} = \frac{a}{4} = N$. We then have $a - 2 = \frac{a}{2}$, or $a = 4$. This implies $x = 2^{16}$, which leaves a remainder of 36 when divided by 100 (use the fact that $2^8 = 256$).

29. Using the identity $\sin^2 x + \cos^2 x = 1$, we square this equation to get $\sin^4 x + \cos^4 x + 2\sin^2 x \cos^2 x = 1$, which implies that $\sin^2 x \cdot \cos^2 x = \frac{2}{9}$. Making the substitutions $a = \sin^2 x$, $b = \cos^2 x$ results in the equations $a + b = 1$, $ab = \frac{2}{9}$, which when solved, yield $(a, b) = \left(\frac{1}{3}, \frac{2}{3}\right)$ or $(a, b) = \left(\frac{2}{3}, \frac{1}{3}\right)$ in either order. Then $\tan^2 x = 2, \frac{1}{2}$, or $\tan x = \sqrt{2}, \frac{\sqrt{2}}{2}$, so the desired sum is $\frac{3\sqrt{2}}{2}$. (Note that $\tan x > 0$, as x lies in the first quadrant.)

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30. The number of sets of possible friendships that Kotori could have with her eight acquaintances is $2^8 = 256$. Within her circle of acquaintances, there are $\binom{8}{2} \cdot \binom{6}{2} \cdot \binom{4}{2} = 2520$ ways to pair them off into good-friend pairs, but this overcounts by 4! (to account for the order in which the pairs are chosen). The total number of graphs Kotori could draw is then $256 \cdot \frac{2520}{24} = 26880 \equiv 880 \bmod 1000$.

