Counting Techniques

- Permutations with repetition: n^r
  - where n is total possibilities, r is how many being chosen
  - Order matters
  - Example: lock combination
- Permutations without repetition: ${}_nP_r = \frac{n!}{(n-r)!}$
  - where n is total possibilities, r is how many being chosen
  - Order and uniqueness matter
  - Example: first, second, and third place winners in a contest
- Permutations with identical repetition: $P = \frac{n!}{k_1!\cdot k_2!\cdots k_i!}$
  - where n is total possibilities, k_i elements of identical type i
  - Order matters but some values are identical
  - Example: all possible 11-letter "words" made with the letters from MISSISSIPPI
- Combinations without repetition: ${}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$
  - where n is total possibilities, r is how many being chosen
  - Uniqueness matters, but order does not
  - Example: lottery numbers
- Combinations with repetition: Stars and Bars technique
  - Neither uniqueness nor order matters
  - Example: distributing a dozen doughnuts to 5 people (not everyone needs to receive a doughnut)

Categorize the following situations with the correct counting technique and then solve.

A. There are 50 students who have applied for the 12-person Student Council at your school. How many distinct councils could you have? $\binom{50}{12} = 1.213 \times 10^{11}$

B. There are 20 entries into the Science Fair at your school. If they award first through fifth place ribbons, how many different possible groups could be given these top 5 awards? ${}_{20}P_5 = 1860480$

C. Missouri's generic license plates (e.g., AA2 B5C) have six positions for which 2 positions are the digits 0-9 and the other 4 positions are letters (A-Z, not including I, O, or Q). The first position represents the letter of the month of renewal. How many distinct license plates are possible? $9 \cdot 23^3 \cdot 10^2 = 10950300$

D. Rachel collects stickers. She currently has 3 sloth stickers, 2 llama stickers, 2 dolphin stickers and 1 cheetah sticker. If she wants to arrange all of the stickers in a line on her notebook, how may different ways can she do this?
$P = \frac{8!}{3!\cdot 2!2!} = 1680$

E. 5 students are seated around a circular cafeteria table. How many different ways can they be seated if rotations do not count as different? $P = (5-1)! = 24$

Example 1: How many ways can I give a dozen doughnuts to 4 people?

<u>Stars & Bars Technique</u>
We could list the donuts that we give to person A as A, donuts to person B as B, and so on. AAAAAABBDDDD would indicate that we give 6 doughnuts to person A, 2 to person B, 0 to person C, and 4 to person D. Yet, we only need to keep track of when we stop giving doughnuts to person A and start giving it to person B. So, we are going to use stars as the doughnuts and a bar to denote when we switch people.

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How would we represent person A getting 0 doughnuts, person B getting 7 doughnuts, person C getting 5 doughnuts, and person D getting 0 doughnuts?

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Both of these are just strings of either stars or bars, so we can use the combination formula of $\binom{n}{r}$ where *n* represents the total number of stars and bars in the string and *r* represents the number of bars.

$$\binom{15}{3} = 455$$

Example 2: How many ways can I give a dozen doughnuts to 4 people if each person gets as least 1 doughnut? (Hint: Give everyone one doughnut and then solve as we did above with the remaining doughnuts.)

After giving everyone 1 doughnut, there are 8 leftover. ****|**|**|

$$\binom{11}{3} = 165$$

Example 3: How many solutions are there to the equation $x_1 + x_2 + x_3 = 10$, where x_1, x_2, x_3 are non-negative integers? (Hint: Think about this just like you are giving out a total of 10 doughnuts to 3 different people.)

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$$\binom{12}{2} = 66$$

Example 4: Your favorite frozen yogurt place has 10 toppings to choose from. On Tuesdays, they allow you to put u 6 toppings on your frozen yogurt cup for the same price. These 6 toppings can be repeated; for example, double peanuts, triple bananas, and hot fudge would work as 6 toppings. How many different combinations of frozen yogurt cups could you make?

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$$\binom{15}{9} = 5005$$

1. William and Jennifer are playing a game. William writes all of the permutations of the letters WILLIAM at the rate of 1 permutation per second, while Jennifer writes all of the permutations of the letters JENNIFER at the rate of 2 permutations per second. Assuming that William and Jennifer start writing at the same time and they don't take any breaks, how many minutes faster will William finish writing? (Mathleague High School State 2016, Target #5)
Answer: 63

2. Compute the number of integers between 100,000 and one million with the property that their digits are distinct and increase from left to right. (ARML 1990, Team #6) Answer: 84

What if this question changed to read "…their digits are non-decreasing from left to right"? How would this change the method used to solve it? Answer: 3003

3. There are 6 people with 15 apples. In how many ways can they distribute the apples amongst themselves so that everyone gets at least 2 apples? (Mathleague High School State 2015, Target #5) Answer: 56

4. How many distinct sequences of 3 letters can be made from the letters in BETTER? (Mathleague Middle State 2017, Sprint #19) Answer: 42

5. How many 3-digit positive integers do not contain the digits 1 or 9? (NYSML 1991, Relay 1-2) Answer: 448

6. Consider the set of 5-digit numbers, each of which is a permutation of the digits 1, 2, 3, 4, 5. There are 120 numbers in this set. Compute the sum of these numbers. (ARML 1992, Team #1) Answer: 3999960

7. The number of different 10-letter "words" that can be made from the letters of the word REASSESSES is the same as the number of different x-letter "words" that can be made from the letters of the word REDUCTIONS. Compute x. (ARML 1993, Team #4) Answer: $x = 4$

8. How many ordered quadruples (a, b, c, d) of positive integers satisfy $a + b + c + d = 20$ and $a, b, c, d \leq 10$? (Mathleague High School State 2016, Sprint #26) Answer: 633

9. Six people of different heights are getting in line to buy donuts. Compute the number of ways they can arrange themselves in line such that no three consecutive people are in increasing order of height, from front to back. (ARML 2015, Individual #10) Answer: 349

10. Consider the equation $x_1 + x_2 + x_3 + x_4 = 15$. How many solutions are there with $2 \leq x_i \leq 5$ for $i \in \{1,2,3,4\}$? Answer: 40