# Rigidity of Convex Polyhedra

03/01/2020

## 1 Heron's formula

### Problem 1

If the three sides of a triangle $\triangle ABC$ have lengths 3cm, 4cm, 5cm, find the area of $\triangle ABC$.

### Problem 2

If the three sides of a triangle $\triangle ABC$ have lengths 4cm, 4cm, 2cm, find the area of $\triangle ABC$.

### Problem 3

If the three sides of a triangle $\triangle ABC$ have lengths $AB = 5cm$, $BC = 6cm$, $CA = 7cm$, find the $\cos \angle ABC$.

## Problem 4

If the three sides of a triangle $\triangle ABC$ have lengths $AB = 5cm$, $BC = 6cm$, $CA = 7cm$, find the area of $\triangle ABC$.

## Problem 5

If the three sides of a triangle $\triangle ABC$ have lengths $AB = c$, $BC = a$, $CA = b$,

1. Find a formula of $\cos\angle ABC$ in term of $a, b, c$.
2. Find a formula of the area of $\triangle ABC$ in term of $a, b, c$.

## 2 Euler's formula

A **graph** $G$ is planar if it can be drawn in the plane $\mathbb{R}^2$ without crossing edges. A graph is simple if there is no multiple edges or loops.

### Problem 6

1. Draw a simple planar graph with 5 vertices, 10 edges.
2. Can you draw a simple planar graph with 5 vertices , 6 edges?

**Euler's formula**. If $G$ is a connected plane graph with $n$ vertices, $e$ edges, and $f$ faces, then

$$n - e + f = 2.$$

The **degree** of a vertex is the number of edges that end in the vertex, where loops count double.

Let $n_i$ denote the number of vertices of degree $i$ in a graph.

### Problem 7

Show that for any planar graph $G$,

$$2e = n_1 + 2n_2 + 3n_3 + 4n_4 + \cdots$$

A $k$-**face** is a face that is bounded by $k$ edges. Let $f_k$ be the number of $k$-faces of a planar graph $G$.

## Problem 8

Show that for any planar graph $G$,

$$2e = f_1 + 2f_2 + 3f_3 + 4f_4 + \cdots$$

## Problem 9

Show that any simple planar graph $G$ with $n > 2$ vertices has at most $3n - 6$ edges.

## Problem 10

Show that any simple planar graph $G$ with $n > 2$ has a vertex of degree at most 5.

## Problem 11

Show that if the edges of $G$ are colored by 2 colors, there is vertex of $G$ with at most two color-changes in the cyclic order of the edges around the vertex.

## 3 Cauchy's rigidity theorem

### Problem 12

For $\triangle ABC$ and $\triangle A'B'C'$, if $AB = A'B'$, $BC = B'C'$, and $\angle ABC \leq \angle A'B'C'$, show that $BC \leq B'C'$.

### Problem 13

If $Q$ and $Q'$ are convex $n$-gons, labeled as in the following figure such that $q_iq_{i+1} = q'_iq'_{i+1}$ holds for the lengths of corresponding edges for $1 \leq i \leq n-1$, and $\alpha_i \leq \alpha'_i$ holds for the sizes of corresponding angels for $2 \leq i \leq n-1$, then the " missing" edge length satisfies

$$q_1q_n \leq q'_1q'_n,$$

with equality if and only if $\alpha_i = \alpha'_i$ holds for all $i$.

### Problem 14

Prove the above property for spherical convex $n$-gons.

A **polyhedron** is a solid in three dimension with flat polygonal faces, straight edges and sharp corners or vertices.

A **convex** polyhedron is the convex hull of finitely many points.

**Assumption:** Assume that $P$ and $P'$ are two convex polyhedra. There is a one to one correspondence between the faces, edges, and vertices of $Q$ and $Q'$ such that corresponding pairs of faces are congruent.

## Problem 15

We color the edges of P as follows: an edge is black if the corresponding interior angle between adjacent facets is larger in $P'$ than in $P$ it is white if the corresponding angle is smaller in $P'$ than in $P$. Prove that if there is at least one pair of vertices $v_1$ and $v_1'$ in $P$ and $P'$ such that the edge angles are not all same, then is one vertex of $v$ that have at most two changes between black and white edges (in cyclic order).

## Problem 16

Look the vertex $v$ and the corresponding $v'$ of $P'$. Intersect $P$ with a small sphere $S_\epsilon$ centered at the vertex $v$, and similarly intersect $P'$ with a sphere $S_\epsilon'$ of the same radius. Find convex spherical polygons $Q$ and $Q'$ such that corresponding arcs have the same lengths.

## Problem 17

Use Problem 15 and 16 to prove the Cauchy rigidity theorem which states that the two polyhedra must be congruent.