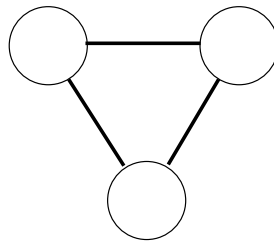


SYMMETRIES AND COUNTING

I. COUNTING “BRACELETS”

Problem 1. (a) How many different ways can you color the circles in this figure

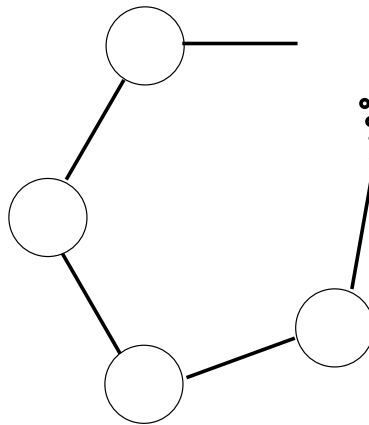


black and white? (There are pages with copies of this figure attached at the end; you may draw them there.)

(b) If we think of the figures as bracelets with 3 beads, which of your “colorings” give the *same bracelet*? How many bracelets are possible?

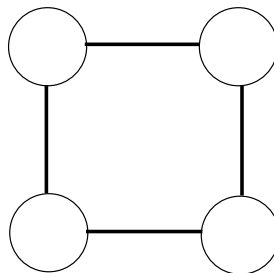
(c) If we use red, green and blue, instead of black and white, how many different ways are there to color the picture? How many *bracelets* are possible? (These should be two different numbers.)

Problem 2. (a) How many ways can you color the circles in a figure like this

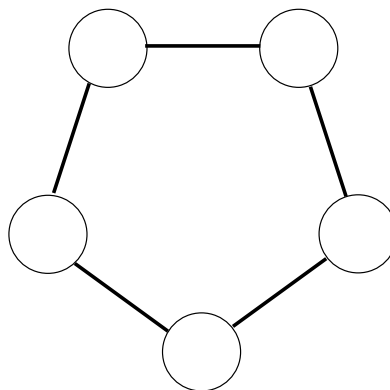


black and white, if there are 4 circles? 5 circles? (N circles?)

(b) With that understood, draw all the possible black and white bracelets with 4 beads . . .

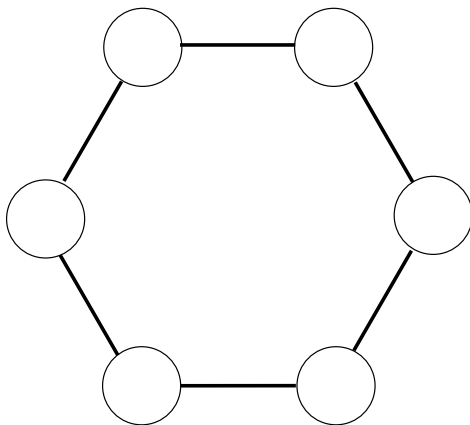


and 5 beads . . .



where two bracelets are “the same” if you can *rotate* one into the other.

Problem 3. Now let's look at bracelets with six beads:



(a) How many bracelets are there if you just equate those which can be rotated on the page into each other?

(b) What if we equate (as we should) bracelets that can be flipped into each other?

How is this different from when we had 3, 4, and 5 beads?

(Feel free to see what you get for 7 beads too!)

II. USING SYMMETRIES TO COUNT

Let's think about bracelets with N beads (in two colors). It gets hard to count these as N gets big, so we'll need another approach. We are going to use a formula, and first I will explain in words where it comes from.

Start by giving names to groups of objects:

- C for all the colorings of the picture with N beads,
- B for the collection of different bracelets, and
- S for all the symmetries.

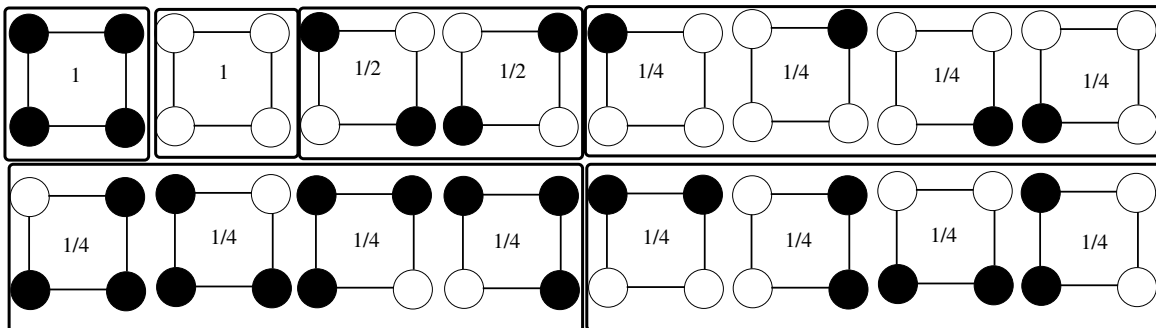
The number of colorings $\#C = 2^N$, as you saw above.

Problem 4. The symmetries include rotations by $360/N$ degrees – once, twice, and so on. They also include the flips (how many?), and the “identity” – the silly symmetry that does nothing. Do we get any more by rotating and then flipping? (Cut out the hexagon on the last page and play with it!) Convince yourself that the number of symmetries $\#S = 2N$.

If s is a symmetry, let $C(s)$ be the colorings that s doesn't change. We say s “fixes” these colorings.

The number $\#B$ of bracelets is what we want to find.

Let $C(c)$ be all the colorings you can get from a given coloring c , by applying symmetries. Then $\frac{1}{\#C(c)}$ is the fraction of the symmetries which fix c . If we add up these fractions over the set $C(c)$, we get $\frac{\#C(c)}{\#C(c)}$, which is 1. Therefore, if we add up these fractions over all colorings, we get the number of bracelets $\#B$. Here is a picture:



On the other hand, $\frac{\#S}{\#C(c)}$ is the number (rather than fraction) of symmetries that fix c . Adding these up gives the number of pairs (c, s) where c is fixed by s , which is the same as adding up all the $\#C(s)$.

So adding up the $\#C(s)$ (for every symmetry s) gives $(\#S) \times (\#B)$, or

$$\#B = \frac{\text{sum of all the } \#C(s)}{\#S}.$$

Problem 5. Using this **counting formula**, recalculate the number of black and white bracelets with 5 beads, then with 7 beads. [Hint: organize the symmetries by “type”: identity, rotation, and flip. The identity fixes all colorings.] Can you do it for N beads, with N any prime number?

What happens if we limit the types of beads that are available?

Problem 6. How many distinct necklaces can be formed using two black, one white and one red bead? (As before, two bracelets are the same if you can rotate or flip one to get the other. All four beads are the same size, and you have to use all of them.) Try it with and without the formula.

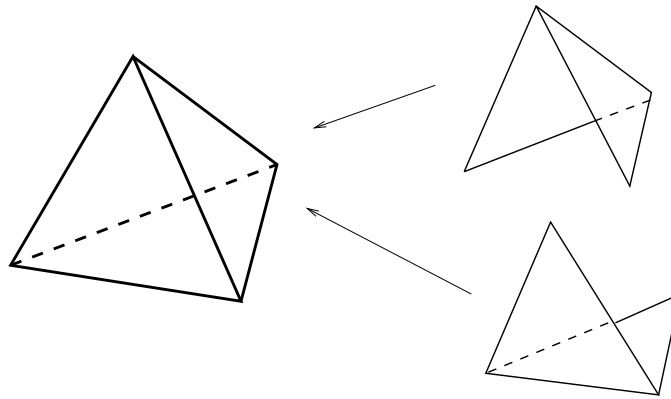
Problem 7. How many (six-bead) necklaces can be formed using three black, two white and one red bead? (Same instructions.)

The same counting formula can be used (how?) to solve the following:

Problem 8. Suppose two identical decks of 52 cards are combined into a big deck. How many three-card hands can be dealt out of the big deck?

III. SYMMETRIES OF TWO SOLIDS

Start by making a tetrahedron out of poster board:



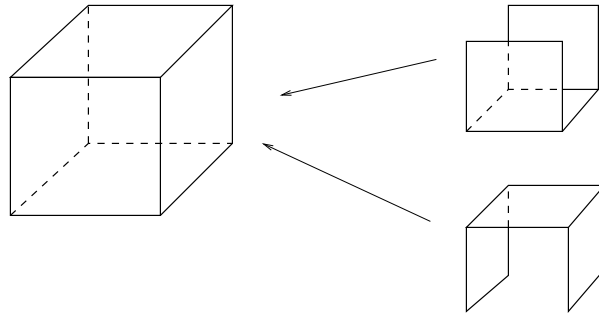
Problem 9. Besides the “identity”, a tetrahedron has 2 different kinds of rotational symmetries: one that you do twice to get back to where you started, and one that you apply 3 times. How many of each do you count? (Altogether you should find $\#S = 12$.)

Problem 10. (a) Now using colored post-it notes, experiment with “coloring” the faces of your tetrahedron in 2 different colors. How many distinct colored tetrahedra do you get? (This should be easy, and give a single-digit number.) (b) Answer the same question for edges. (This is much more interesting. Why? How many different ways are there to paint 3 edges black?)

Problem 11. The counting formula works in many situations, not just the “bracelet problem” on the previous pages. Use it to compute the number of rotationally distinct edge colorings in 2 colors; then try it in 3 colors.

A cube has 6 faces, just like the tetrahedron has 6 edges. But the number of cube-face colorings is different from the number of tetrahedron-edge colorings.

Make a cube out of poster board:



Problem 12. What are the rotational symmetries of a cube? [Hint: besides the identity, you should find 4 different “types”. How many of each type are there? The total should be 24.]

Problem 13. (a) How many ways are there to color the cube with 2 colors? (Try it first, then see if you can use the formula.)

(b) 3 colors? (For this, you definitely need the formula.)

IV. SYMMETRIES OF A CRYSTAL

Almost a century before X-rays could be used to look “inside” crystals to see their atomic lattice structure, 19th Century mathematicians figured out what configurations were possible, by narrowing down the possible symmetries. It turns out that there is an easy way to do this using our counting formula! (This is significantly more challenging than what we’ve already done, but it’s worth trying if you feel comfortable so far.)

If S is the set of all (say, N) rotational symmetries of some 3-dimensional object (of any kind), look at all the axes of these rotations and intersect them with the surface of a ball. This gives a set of points on the ball, and we can group the points that are rotated into one another by S .¹ Call the set of points C , and the set of groups of points B . We know that

$$\#B = \frac{\text{sum of all the } \#C(s)}{\#S},$$

which is easy to use here: $\#C(\textit{identity}) = \#C$, and $\#C(\text{any other } s) = 2$ (the two points where the axis of rotation for s hit the ball). So

$$\#B = \frac{\#C + 2(N - 1)}{\#S} = \frac{\#C - 2}{N} + 2,$$

where I used that $\#S = N$.

We also know that a fraction, say $m = \frac{N}{n}$, of the symmetries fix any given point, and that then n is the number of points in its “group”. But then $n = \frac{N}{m}$ and $\#C$ is a sum of the form $\frac{N}{m_1} + \dots + \frac{N}{m_r}$, where $r = \#B$ and each term of the sum is the number of points in one of the groups. So we have

$$r = \frac{1}{m_1} + \dots + \frac{1}{m_r} + 2 - \frac{2}{N},$$

where the m_i are numbers dividing N . Each of them is at least 2, which leads to the inequality

$$r \leq \frac{r}{2} + 2 - \frac{2}{N}$$

¹It turns out that S cannot take a point in this set to a point not in this set.

or

$$r \leq 4 - \frac{4}{N}$$

which means r has to be 2 or 3.

If $r = 2$, then you get $\frac{1}{m_1} + \frac{1}{m_2} = \frac{2}{N}$, which forces $N = m_1 = m_2$, so that every symmetry fixes every point in C – i.e., they are all rotations about one axis (boring).

So the interesting case is $r = 3$.

Problem 14. Your turn: what are all the possibilities for (m_1, m_2, m_3) (and N) when $r = 3$?

From the fact that m_i is the number of rotations fixing some axis, you know that it is actually the number of rotations through that axis that returns you to the start.

Problem 15. For each of your possibilities from Problem 11, use this observation to describe the set of symmetries geometrically. You should get the symmetries of a bracelet with any number of beads, of a cube, and of a tetrahedron! There is also one mysterious left over case: $(2, 3, 5)$, which corresponds to the 60 symmetries of an icosahedron.

It is possible to further refine this classification if we insist that the 3-dimensional object is a crystal, and then you get only finitely many possibilities (no icosahedron, no bracelets with 5 or ≥ 7 beads, etc.), but we'll stop here.

Challenge: Consider the dodecahedron (left) and icosahedron (right)



which are *dual* in the sense that faces and vertices are in one-to-one correspondence and the rotational symmetries are the same.

Problem 16. Enumerate these rotational symmetries. (In addition to the “identity” which does nothing, you should find rotations by multiples of 72° , 120° , and 180° for a total of 60 overall.)

Problem 17. Find the number of black and white colorings of the 12 faces of the dodecahedron (or equivalently, the 12 vertices of the icosahedron). Then try it with the faces of the icosahedron. You should use a version of the counting formula just before Problem 5.

